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TRANSCENDENCE PROPERTIES OF CERTAIN QUANTITIES OVER THE QUOTIENT FIELD $\mathbb{F}_q[x]$

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Transcendence properties of certain quantities over the quotient field of $\mathbb{F}_q[x]$

by

J.M. Geijsel

ABSTRACT

In this report the result of I. Wade:
"the zeros of the Carlitz-ψ-function and the values $\psi(\alpha)$, where $\alpha \neq 0$, α algebraic over $\mathbb{F}_q[x]$, are transcendental over $\mathbb{F}_q[x]$", is generalized to a larger class of functions. The proof uses a combinatorial lemma and Schneider's method.

As a special case we have the result:
"The zeros of the Carlitz-Besselfunctions $J_n(t)$ are transcendental over $\mathbb{F}_q[x]$ and at least one of the two values $J_n(\alpha)$, $J_n(x\alpha) - xJ_n(\alpha)$ for $\alpha \neq 0$, α algebraic, is transcendental over $\mathbb{F}_q[x]$."

KEY WORDS & PHRASES: *Transcendency, Carlitzfunctions.*

1. NOTATION AND HISTORICAL INTRODUCTION

Let $\mathbb{F}_q$ be a finite field with $q = p^{n_0}$ (p prime) elements. We denote by $\mathbb{F}_q[x]$ the ring of polynomials with coefficients in $\mathbb{F}_q$ and by $\mathbb{F}_q\{x\}$ its quotientfield.

For $0 \neq E \in \mathbb{F}_q[x]$ we define the (logarithmic) valuation

$$\mathrm{dg}E = \text{degree of } E$$

and

$$\mathrm{dg}0 = -\infty.$$

For $Q \in \mathbb{F}_q\{x\}$ where $Q = \frac{E}{F}$ with $E,F \in \mathbb{F}_q[x]$ and $F \neq 0$ we define

$$\mathrm{dg}Q = \mathrm{dg}E - \mathrm{dg}F.$$

The completion of $\mathbb{F}_q\{x\}$ with respect to this valuation is denoted by $\mathcal{F}$.

The valuation on $\mathcal{F}$ can be extended to the algebraic closure of $\mathcal{F}$ and the completion of this algebraic closure is denoted by Φ. The valuation of Φ also will be denoted by dg.

1.1. <u>DEFINITION</u>

$$F_0 = 1;\ F_k = \prod_{j=0}^{k-1} (x^{q^k} - x^{q^j}),\ k > 0; \qquad F_k^{-1} = 0,\ k < 0.$$

$$L_0 = 1;\ L_k = \prod_{j=1}^{k} (x^{q^j} - x),\ k > 0.$$

$$\psi(t) := \sum_{k=0}^{\infty} (-1)^k \frac{t^{q^k}}{F_k}.$$

The function $\psi(t)$ was introduced in a paper of L. CARLITZ [1], where he showed, among other things, the following: The function $\psi(t)$ has zero's in the points $E\xi$, with $E \in \mathbb{F}_q[x]$ and where

$$\xi = \lim_{k\to\infty} \frac{(x^q-x)^{\frac{q^k}{q-1}}}{L_k} .$$

These are the only zero's of $\psi(t)$.

Remark that $dg\xi = \frac{q}{q-1}$.

In 1941 L.I. WADE [8] proved the transcendency over $\mathbb{F}_q\{x\}$ of the quantities

(i) ξ

(ii) $\psi(\alpha)$ for $\alpha \neq 0$, α algebraic over $\mathbb{F}_q\{x\}$

(iii) $\sum_{k=0}^{\infty} c_k \frac{E^{q^k}}{F_k}$ where $c_k \in \mathbb{F}_q$, $c_k \neq 0$ for infinitely many k, and where $E \in \mathbb{F}_q[x]$.

In this paper we prove the following results:

(A1) Let $\eta \neq 0$ be a zero of the function

$$f(t) = \sum_{k=0}^{\infty} c_k \frac{t^{q^k}}{F_k},$$

where $c_k \in \mathbb{F}_q$ and $c_k \neq 0$ for infinitely many k, then η is transcendental over $\mathbb{F}_q\{x\}$.

(A2) Let $\eta \neq 0$ be a zero of the (Carlitz-Bessel) function

$$J_n(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{q^{n+k}}}{F_{n+k}F_k^{q^n}}, \quad n \in \mathbb{Z}$$

then η is transcendental over $\mathbb{F}_q\{x\}$.

(B) Suppose $\alpha \in \Phi$, $\alpha \neq 0$, α algebraic over $\mathbb{F}_q\{x\}$ and let

$$f(t) = \sum_{k=0}^{\infty} c_k \frac{t^{q^k}}{F_k},$$

where $c_k \in \mathbb{F}_q$ and there exists a $c \in \mathbb{F}_q$, $c \neq 0$ such that

$$c_{k+1} = c.c_k, \quad k = 0,1,2,\ldots,$$

then $f(\alpha)$ is transcendental over $\mathbb{F}_q\{x\}$.

(C) Suppose $\alpha \in \Phi$, $\alpha \neq 0$, α algebraic over $\mathbb{F}_q\{x\}$ and let

$$J_n(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{q^{n+k}}}{F_{n+k}F_k^{q^n}}, \quad n \in \mathbb{Z}$$

then at least one of the elements $\{J_n(\alpha), \Delta J_n(\alpha)\}$, where $\Delta J_n(\alpha) = J_n(x\alpha) - xJ_n(\alpha)$, is transcendental over $\mathbb{F}_q\{x\}$.

The proofs of the results B and C use a refinement of the methods in [9] and in [5] and the results A1 respectively A2. These results will be proved in §4 and §5.

2. THE ZEROS OF $\psi(t)$, $f(t)$ AND $J_n(t)$.

The field Φ is complete with respect to the discrete valuation dg. We use Newton's method to determine the size of the roots of a polynomial over Φ. (See [10]).

Let $g(x) = a_0+a_1t+a_2t^2+\ldots+a_nt^n$ be a polynomial over Φ and assume $a_0, a_n \neq 0$.

In the two dimensional Euclidean space we associate a polygon with the polynomial g as follows:

if $a_i \neq 0$ take the point $(i, -dga_i)$,

if $a_i = 0$ then $dga_i = -\infty$ and no point of the space belongs to a_i.

The lower convex envelope of the set of points $\{(i,-dga_i) \mid i=0,1,\ldots,n\}$ is called the *Newton polygon* of g (with respect to dg).

2.1. <u>LEMMA</u>. *Let* $g(t) = a_0+a_1t+\ldots+a_nt^n$ *with* $a_0, a_n \neq 0$ *be a polynomial over* Φ. *Suppose that* $(r,-dga_r) \leftrightarrow (s,-dga_s)$ *with* $s > r$ *is any segment of the Newton polygon of* g *and its slope is* m. *Then* g *has exactly* s-r *roots* $\beta \in \Phi$ *with* $dg\beta = m$.

Let

$$R_1 := \min_{i>0} \frac{dga_0 - dga_i}{i}$$

$$i_1 := \max \{i \mid \frac{dga_0-dga_i}{i} = R_1\}$$

and inductively if the set $\{i \mid i_{k-1} < i \leq n\}$ *is not empty,*

$$R_k := \min_{i>i_{k-1}} \frac{dga_{i_{k-1}} - dga_i}{i-i_{k-1}},$$

$$i_k := \max \{i \mid \frac{dga_{i_{k-1}} - dga_i}{i-i_{k-1}} = R_k\}.$$

Then g *has exactly* i_1 *zeros* β *with* $dg\beta = R_1, \ldots, i_k - i_{k-1}$ *zeros* β *with* $dg\beta = R_k$. *Furthermore* $R_1 < R_2 < \ldots < R_k$.

<u>PROOF</u>. We recall the proof of WEISS [10], prop. 3-1-1, in our notation. We may suppose that $a_0 = 1$. Let $\beta_1, \ldots, \beta_n \in \Phi$ be the zeros of g, ordered in such a way that

$$\begin{aligned}
dg\beta_1 &= \ldots = dg\beta_{s_1} = m_1\\
dg\beta_{s_1+1} &= \ldots = dg\beta_{s_2} = m_2\\
&\vdots\\
dg\beta_{s_i+1} &= \ldots = dg\beta_n = m_{i+1} \text{ with } m_1<m_2<\ldots<m_{i+1}.
\end{aligned}$$

Using the symmetric polynomials in the roots $\beta_1^{-1}, \ldots, \beta_n^{-1}$ of $a_0t^{-n}+\ldots+a_{n-1}t^{-1}+a_n$ we get:

$$dga_0 = dg1 = 0$$

$$dga_1 = dg \sum_{i=1}^{n} \beta_i^{-1} \le \max_{1\le i\le n} (-dg\beta_i) = -m_1$$

$$dga_2 = dg \sum_{\substack{i,j=1\\ i\neq j}}^{n} \beta_i^{-1}\beta_j^{-1} \le \max_{\substack{1\le i,j\le n\\ i\neq j}} (-dg\beta_i\beta_j) = -2m_1$$

$$\vdots$$

$$dga_{s_1} = dg \sum_{j_k\neq j_l} \beta_{j_1}^{-1}\beta_{j_2}^{-1}\cdots\beta_{j_{s_1}}^{-1} = -s_1m_1, \text{ since this elementary}$$

symmetric function has exactly one maximal term $\beta_1^{-1}\beta_2^{-1}\cdots\beta_{s_1}^{-1}$.

$$dga_{s_1+1} \le -s_1m_1 - m_2$$

$$\vdots$$

$$dga_{s_2} = -s_1m_1 - (s_2-s_1)m_2,$$

and this process is continued in the obvious way.

Now we can form the Newton polygon of g, which consists of the segments:

$$(0,0) \leftrightarrow (s_1,s_1m_1)$$
$$(s_1,s_1m_1) \leftrightarrow (s_2,s_1m_1+(s_2-s_1)m_2)$$
$$\vdots$$
$$(s_i,s_1m_1+(s_2-s_1)m_2+\ldots+(s_i-s_{i-1})m_i) \leftrightarrow (n,s_1m_1+(s_2-s_1)m_2+\ldots$$
$$\ldots+(n-s_i)m_{i+1}).$$

The slopes of the segments are respectively

$$\frac{s_1m_1-0}{s_1-0} = m_1, \quad \frac{s_1m_1+(s_2-s_1)m_2-s_1m_1}{s_2-s_1} = m_2,\ldots,m_{i+1}.$$

From the inequalities for $dga_0,\ldots,dga_n$ it follows that

$$\frac{dga_0-dga_i}{i} \geq m_1 \text{ for } i = 1,2,\ldots,s_1,$$

$$\frac{dga_0-dga_i}{i} > m_1 \text{ for } i = s_1+1,\ldots,n$$

and

$$\frac{dga_0-dga_{s_1}}{s_1} = m_1.$$

Hence $R_1 = m_1$ and $i_1 = s_1$, which means: g has exactly i_1 roots β with $dg\beta = R_1$.

It also follows from the inequalities above that

$$\frac{dga_{i_1}-dga_i}{i-i_1} = \frac{dga_{s_1}-dga_i}{i-s_1} \geq \min_{1\leq j\leq s_2-s_1} \frac{-s_1m_1+(s_1m_1+jm_2)}{s_1+j-s_1} = m_2$$

$$\text{for } i = s_1+1,\ldots,s_2$$

and

$$\frac{dga_{i_1}-dga_i}{i-i_1} > m_2 \text{ for } i = s_2+1,\ldots,n,$$

while

$$\frac{dga_{i_1}-dga_{s_2}}{s_2-i_1} = m_2.$$

Hence $R_2 = m_2$ and $i_2 = s_2$ which means: g has exactly i_2-i_1 roots β with $dg\beta = R_2$.

The lemma now follows by proceeding in the obvious way.

REMARK. Since we had ordered the zeros in such a way that $m_1<m_2<\ldots<m_{i+1}$ we find $R_1<R_2<\ldots<R_{i+1}$. □

In his thesis W. SCHÖBE [7], II §3 proved the more general result:

2.2. THEOREM. *The function* $g(t) = a_h t^h + a_{h+1} t^{h+1} + \ldots$ *with* $a_i \in \Phi$, $i=h,h+1,\ldots$; $a_h \neq 0$; $h \geq 0$ *has*

(i) *a zero of order* h *in* $t = 0$.

(ii) i_1-h *zeros* β *in* Φ *with* $dg\beta = R_1$, *where*

$$R_1 = \min_{i>h} \frac{dga_h - dga_i}{i-h},$$

if this minimum exists,

and

$$i_1 = \max_{i>h}\{i \mid \frac{dga_h - dga_i}{i-h} = R_1\},$$

if this maximum exists.

(iii) i_k-i_{k-1} *zeros* β *in* Φ *with* $dg\beta = R_k$ $(k\geq 2)$ *where,*

$$R_k = \min_{i>i_{k-1}} \frac{dga_{i_{k-1}} - dga_i}{i-i_{k-1}},$$

if this minimum exists

and

$$i_k = \max_{i>i_{k-1}} \{i \mid \frac{dga_{i_{k-1}} - dga_i}{i-i_{k-1}} = R_k\},$$

if this maximum exists.

These are the only zeros of g.

Our proof of theorem 2.2 will be based on several lemmas.

2.3. LEMMA. *Let* g *be defined as in lemma 2.2. Let* $R = -\limsup_{i\to\infty} \frac{dga_i}{i}$ *be the radius of convergence of* g. *If* R_k *as defined in lemma 2.2 exists,* *then* $R_k \le R$.

PROOF. Let $\varepsilon > 0$ be arbitrarily small. Since R is the radius of convergence of g for infinitely many i we have

$$\frac{dga_i}{i} > -R-\varepsilon.$$

From the existence of R_k it follows that for all $i > i_{k-1}$

$$R_k \le \frac{dga_{i_{k-1}} - dga_i}{i-i_{k-1}}.$$

We now have that for infinitely many $i > i_{k-1}$

$$R_k < \frac{dga_{i_{k-1}}}{i-i_{k-1}} + \frac{i(R+\varepsilon)}{i-i_{k-1}}.$$

Since $\lim_{i\to\infty} \frac{dga_{i_{k-1}}}{i-i_{k-1}} = 0$ and $\lim_{i\to\infty} \frac{i(R+\varepsilon)}{i-i_{k-1}} = R + \varepsilon$ we get

$$R_k \le R + \varepsilon.$$

Since ε can be chosen arbitrarily small it follows that

$$R_k \le R. \qquad \square$$

2.4. LEMMA. *Let* g *be defined as in lemma* 2.2. *If* R_k *exists and either* $R_k < R$ *or* $g(t)$ *converges for a certain* t *with* $dgt = R$, *then there exists an index* i_k *such that*

$$\frac{dga_{i_{k-1}} - dga_i}{i-i_{k-1}} > R_k, \quad i > i_k$$

and

$$\frac{dga_{i_{k-1}} - dga_i}{i-i_{k-1}} \le R_k, \quad i_{k-1} < i \le i_k,$$

where

$$\frac{dga_{i_{k-1}} - dga_{i_k}}{i_k - i_{k-1}} = i_k.$$

If $k = 1$ *we must read* h *instead of* i_{k-1}.

PROOF. Suppose R_1 exists. Since $R_1 \le R$ according to lemma 2.3 we have: $\forall N \in \mathbb{N}\ \exists i_0 \in \mathbb{N}$ such that

$$dga_i + idgt < -N,\ i > i_0,$$

for those t with $dgt \le R_1$ for which g converges. (*)

Now choose N such that

$$dga_h + hR_1 > -N$$

(this is possible since $a_h \neq 0$ and $R_1 \in \mathbb{R}$). Suppose there exists a monotonically increasing sequence $(i_j)_{j=1}^{\infty}$ such that

$$\frac{dga_h - dga_{i_j}}{i_j - h} = R_1 \text{ for } j = 1,2,\ldots \qquad \text{(i.e. } i_1 \text{ does not exist)},$$

then

$$dga_h + hR_1 = dga_{i_j} + i_j R_1 \quad \text{for } j = 1,2,\ldots\ .$$

Hence if $i_j > i_0$ it follows from (*) that if we take a t with $dgt = R_1$ for which g converges then

$$dga_h + hR_1 < -N.$$

This is in contradiction with the choice of N.

For $k > 1$ the proof is the same. (Note that $a_{i_{k-1}} \neq 0$). □

REMARK.

(i) First we note that if there exists a $t \in \Phi$ with $dgt = R_1$ for which g converges then g converges for every t with $dgt = R_1$.

(ii) If $R_k = R$ and $g(t)$ does not converge for any t with $dgt = R$ then we want to know if $g(t)$ can have any zeros $\neq 0$ with $R_{k-1} < dgt < R_k$. We shall prove that $g(t) \neq 0$ for all $t \neq 0$ with $R_{k-1} < dgt < R_k$.

It is sufficient to consider the case $k = 1$; then $R_1 = R$ and $g(t)$ does not converge for any t with $dgt = R$. Suppose $g(u) = 0$, $u \neq 0$ and $dgu < R$. For all n the polynomial $P_n(t) = a_0+a_1t+\ldots+a_nt^n$ has no zeros β with $dg\beta < R$. Now let N be such that $dga_0 > -N$, then choose n such that $dg(g(t)-P_n(t)) < -N$ for all t with $dgt \leq dgu(<R)$. Then

$$dg(P_n(u)) = dg(g(u)-P_n(u)) < -N. \qquad (*)$$

On the other hand

$$dga_0 \geq dga_i + idgu \text{ for all } i \geq 0$$

and

$$dga_0 > dga_i + idgu \text{ for all } i > 0,$$

therefore

$$dgP_n(u) = \max(dga_i+idgu) = dga_0,$$

which contradicts $(*)$.

Conclusion: If $R_1 = R$ and $g(t)$ does not converge for any t with $dgt = R$ then $g(t)$ has no zeros except possibly 0.

<u>EXAMPLE</u>. $\lambda(t) := \sum_{j=0}^{\infty} \frac{t^{q^j}}{L_j}$ is the inverse of $\psi(t)$. (See [1] th. 7.1). Now $R_1 = R = \frac{q}{q-1}$ and i_1 does not exist. Furthermore $\lambda(t)$ does not converge for all t with $dgt = \frac{q}{q-1}$. Hence λ has no zeros β with $dg\beta < \frac{q}{q-1}$ except $\beta = 0$.

2.5. <u>LEMMA</u>. *Suppose* $g(t) = a_0+a_1t+\ldots$ *has radius of convergence* $R > -\infty$ *and suppose* $R_1(g) = R_1$ *exists. If* t_0 *is a zero of* g *with* $dgt_0 = R_1$ *then there exists a* $k \in \mathbb{N} \cup \{0\}$ *such that* $h_\kappa(t) = \dfrac{g(t)}{(t-t_0)^{q^\kappa}}$ *has a zero in* t_0 *for* $\kappa = 0,1,2,\ldots,k-1$ *while* $h_k(t_0) \neq 0$.

<u>PROOF</u>. Choose $k \in \mathbb{N} \cup \{0\}$ such that $q^k > i_1$. The function

$$h_k(t) = \frac{g(t)}{(t-t_0)^{q^k}} = b_0 + b_1t + b_2t^2 + \ldots$$

has radius of convergence R. Since $q^k > i_1$ we have

(1) $$dga_\nu + \nu R_1 < dga_0, \text{ for } \nu > i_1.$$

Especially

$$dga_{q^k} + q^k R_1 < dga_0.$$

But

$$a_0 = -b_0 t_0^{q^k},$$

hence

(2) $$dgb_0 = dga_0 - q^k R_1.$$

Furthermore

$$a_{q^k} = b_0 - b_{q^k} t_0^{q^k},$$

and therefore

$$dgb_{q^k} + q^k R_1 \leq \max(dga_{q^k}, dgb_0).$$

According to (1) and (2) we get

(3) $$dgb_{q^k} + q^k R_1 = dgb_0.$$

From the definition of $h_k(t)$ it follows that h_k has no zeros in $\{t \mid dgt < R_1\}$. Therefore, if $R_1(h_k)$ does exist then

$$\min_{i>0} \frac{dgb_0 - dgb_i}{i} \geq R_1 = R_1(g).$$

However, since $\frac{dgb_0 - dgb_{q^k}}{q^k} = R_1$ we have $R_1(h_k) = R_1$.

Furthermore we have

$$a_{2q^k} = b_{q^k} - b_{2q^k} t_0^{q^k}$$

and therefore

$$dgb_{2q^k} + q^k R_1 \le \max(dga_{2q^k}, dgb_{q^k})$$

or

$$dgb_{2q^k} + 2q^k R_1 \le \max(dga_{2q^k} + q^k R_1, dgb_{q^k} + q^k R_1).$$

Using (1), (2) and (3) we find

$$dgb_{2q^k} + 2q^k R_1 = dgb_0.$$

By proceeding in the obvious way we find

$$dgb_{nq^k} + nq^k R_1 = dgb_0 \qquad \text{for } n = 1,2,\ldots .$$

Hence $i_1(h_k)$ does not exist, which contradicts lemma 2.4. □

2.6. LEMMA. *If* R_1 *does not exist then* g *converges for* $t = 0$ *only.*

PROOF. Suppose R_1 does not exist, i.e. there exists a sequence (a_{i_j}) such that

$\forall N \in \mathbb{N}\ \exists j_0 = j_0(N)$ such that for $j > j_0$

$$\frac{dga_0 - dga_{i_j}}{i_j} < -N.$$

Suppose g converges at the point t, then

$\forall N \in \mathbb{N}\ \exists i_0 = i_0(N)$ such that for $i > i_0$

$$dga_i + i\,dgt < -N.$$

Now choose $N \in \mathbb{N}$ then there exists $j^* > j_0, i_0$ such that for $j > j^*$,

$$Ni_j + i_j dgt < dga_{i_j} - dga_0 + i_j dgt < -N - dga_0$$

or

$$dgt < \frac{-N(i_j+1)}{i_j} - dga_0.$$

Hence

$$dgt \le \lim_{j\to\infty} \frac{-N(i_j+1)}{i_j} - dga_0 = -N - dga_0.$$

Since N was chosen arbitrarily it follows that $dgt = -\infty$, which means $t = 0$. □

2.7. LEMMA. *Suppose* R_1 *exists and* $g(t)$ *converges for all* t *with* $dgt = R_1$. *Then* $g(t)$ *has at least one zero* $\neq 0$ *with* $dgt \le R_1$.

PROOF. We may suppose that $h = 0$, then

$$g(t) = a_0 + a_1 t + a_2 t^2 + \ldots .$$

Suppose $g(t)$ has no zeros t with $dgt \le R_1$. Then $\exists N_0 \in \mathbb{N}$ such that for all t with $dgt \le R_1$

$$dg\, g(t) > -N_0 \qquad (*).$$

Since $g(t)$ converges for all t with $dgt \le R_1$ we have

(a) $\forall N \in \mathbb{N}\ \exists i_0 \in \mathbb{N}$ such that for $i > i_0(N)$ and $dgt \le R_1$

$$dga_i + idgt < -N$$

(b) according to lemma 2.4 i_1 does exist.

Now choose $N > N_0$ and $n > \max(i_0, i_1)$ then

$$\min_{0<i\le n} \frac{dga_0 - dga_i}{i} = R_1.$$

We write

$$g(t) = a_0 + a_1 t + \ldots + a_n t^n + g^*(t).$$

Then

$$dg\, g(t) \le \max(dg(a_0 + a_1 t + \ldots + a_n t^n), dg\, g^*(t))$$

and for all t with $dgt \le R_1$

$$\mathrm{dg}\, g^*(t) \le \max_{i>n}(\mathrm{dg}a_i + i\,\mathrm{dg}t) < -N.$$

According to lemma 2.1

$$P_N(t) = a_0 + a_1 t + \ldots + a_n t^n$$

has exactly i_1 zeros β with $\mathrm{dg}\beta = R_1$.

Let β be a zero of $P_N(t)$ with $\mathrm{dg}\beta = R_1$, then

$$\mathrm{dg}\, g(\beta) = \mathrm{dg}\, g^*(\beta) < -N < -N_0.$$

This contradicts (*).

Since $g(0) = a_0 \neq 0$ it follows that $g(t)$ has at least one zero $\neq 0$ in $\{t \mid \mathrm{dg}t \le R_1\}$. □

2.8. <u>LEMMA</u>. *Suppose* R_1 *exists and* $g(t)$ *converges for all* t *with* $\mathrm{dg}t = R_1$. *Then* $g(t)$ *has exactly* $i_1 - h$ *zeros* β *with* $\mathrm{dg}\beta = R_1$.

<u>PROOF</u>. Again we may suppose that $h = 0$.

$$g(t) = a_0 + a_1 t + \ldots + a_n t^n + \ldots$$

(i) First we prove that $g(t)$ has at most i_1 different zeros β with $\mathrm{dg}\beta = R_1$. For all t with $\mathrm{dg}t \le R_1$ we have
$\forall N \in \mathbb{N}\ \exists i_0 \in \mathbb{N}$ such that for $i > i_0$

$$\mathrm{dg}a_i + i\,\mathrm{dg}t < -N.$$

According to lemma 2.4 i_1 does exist.

Choose $n > \max(i_0, i_1)$ and define

$$P_N(t) = a_0 + a_1 t + \ldots + a_n t^n,$$

$$g(t) = P_N(t) + g^*(t).$$

According to lemma 2.1 $P_N(t)$ has exactly i_1 zeros $\beta_{N1}, \ldots, \beta_{Ni_1}$ such that $\mathrm{dg}\beta_{Nj} = R_1$, $j = 1, \ldots, i_1$. Let $\beta_{N1}, \ldots, \beta_{Ni_1^*}$ be all the different ones among them. Note that $1 \le i_1^* \le i_1$.

$$P_N(t) = a \prod_{j=1}^{i_1} (t - \beta_{Nj}) \prod_{\nu=i_1+1}^{n} \left(1 - \frac{t}{\beta_{N\nu}}\right),$$

where

$$a = \frac{(-1)^{i_1} a_0}{\prod_{j=1}^{i_1} \beta_{Nj}} .$$

Remark that $dga = dga_0 - i_1R_1$ and that a does not depend on N. Furthermore, $dg\beta_{N\nu} > R_1$ for $\nu = i_1+1,\ldots,n$.

For $dgt \le R_1$ we then have:

$$dgP_N(t) = dga + dg \prod_{j=1}^{i_1} (t-\beta_{Nj}) = dga_0 - i_1R_1 + \sum_{j=1}^{i_1} dg(t-\beta_{Nj}),$$

or

$$\sum_{j=1}^{i_1} dg(t-\beta_{Nj}) = dgP_N(t) - dga_0 + i_1R_1. \qquad (*)$$

Suppose $g(t)$ has more than i_1^* different zeros t with $dgt \le R_1$; say $t_1,t_2,\ldots,t_m$ are m zeros with $dgt_i = R_1$, where $m > i_1^*$. Then define

$$\lambda := \min_{\substack{i \ne j \\ 1 \le i,j \le m}} dg(t_i-t_j).$$

Let

$$U_{\lambda j} = \{t \mid dg(t-t_j) < \lambda\}.$$

Then the sets $U_{\lambda 1},\ldots,U_{\lambda m}$ are disjoint.

According to (*) we have

$$\sum_{j=1}^{i_1} dg(t_1-\beta_{Nj}) = dgP_N(t_1) - dga_0 + i_1R_1.$$

Since $g(t_1) = 0$, $dgP_N(t_1) < -N$. Hence there exists at least one $j_1 \in \{1,2,\ldots,i_1^*\}$ such that

$$dg(t_1-\beta_{Nj_1}) < \frac{-N-dga_0+i_1R_1}{i_1}$$

and furthermore it follows (for N large enough):

$$dgt_1 = dg\beta_{Nj_1} = R_1.$$

Now let N be such that

$$\frac{-N-dga_0+i_1R_1}{i_1} < \lambda.$$

Then $\beta_{Nj_1} \in U_{\lambda 1}$. Hence for the zero t_1 there is at least one zero β_{Nj_1} in $U_{\lambda 1}$, and analogously for the zeros $t_2,\dots,t_m$ there is at least one zero β_{Nj_m} of $P_N(t)$ in $U_{\lambda 2},\dots$, resp. $U_{\lambda m}$. Since the $U_{\lambda j}$ are disjoint there must be at least m different β_{Nj}, which leads to a contradiction.

Hence g(t) has at most $i_1^* \le i_1$ different zeros t with dgt = R_1.

(ii) Finally we must show that g(t) has exactly i_1 zeros t with dgt = R_1. It is obvious from lemma 2.5 that if t_0 is a zero of g(t) with $dgt_0 = R_1$ then there exists a $\mu \in \mathbb{N}$ such that

$$h_i(t) = \frac{g(t)}{(t-t_0)^i}$$

has a zero in $t = t_0$ for $i = 1,\dots,\mu-1$, but $h_\mu(t_0) \neq 0$; the natural number μ is called the multiplicity of the zero t_0.

Let $T = \{\beta_1,\beta_2,\dots,\beta_k\}$ be the set of all zeros of g(t) with $dg\beta_i = R_1$, where

$$\beta_1 = \beta_2 = \dots = \beta_{\mu_1} = t_1$$
$$\beta_{\mu_1+1} = \dots = \beta_{\mu_1+\mu_2} = t_2$$
$$\vdots$$
$$\beta_{\mu_1+\dots+\mu_{m-1}} = \dots = \beta_{\mu_1+\dots+\mu_m} = t_m$$

and μ_i is the multiplicity of t_i $(i=1,\dots,m)$ and $k = \mu_1+\dots+\mu_m$.

Define

$$h(t) := \frac{g(t)}{(t-\beta_1)(t-\beta_2)\dots(t-\beta_k)} .$$

Since the radius of convergence of $\frac{g(t)}{t-\beta_1} = \frac{g(t)-g(\beta_1)}{t-\beta_1}$ is also R, by repeating this argument we find that h(t) has radius of convergence R.

Define

$$Q(t) := (t-\beta_1)\ldots(t-\beta_k).$$

Now we write

$$Q(t) = \sum_{\nu=0}^{k} d_\nu t^\nu$$

and

$$h(t) = \sum_{\nu=0}^{\infty} b_\nu t^\nu .$$

Since $h(t)$ has no zeros in $\{t \mid \mathrm{dg}t \leq R_1\}$ we have according to *(i)*:

(a) $$\mathrm{dg}b_0 > \mathrm{dg}b_i + iR_1 \quad \text{for } i = 1,2,\ldots \quad .$$

According to lemma 2.1 we have

$$k = \max\{i \mid \frac{\mathrm{dg}d_0 - \mathrm{dg}d_i}{i} = R_1\},$$

hence

(b) $$\mathrm{dg}d_i + R_1 i \leq \mathrm{dg}d_0 = \mathrm{dg}d_k + R_1 k, \quad \text{for } i = 0,1,\ldots,k.$$

Now from $g(t) = Q(t).h(t)$ it follows that

$$a_i = \sum_{j=0}^{k} d_j b_{i-j}, \qquad i = 0,1,\ldots,$$

and therefore for each t with $\mathrm{dg}t = R_1$ we have:

$$\mathrm{dg}a_i + iR_1 = \mathrm{dg}a_i t^i = \mathrm{dg}(\sum_{j=0}^{k} d_j t^j b_{i-j} t^{i-j}) \leq$$

$$\leq \max_{0 \leq j \leq k} (\mathrm{dg}d_j t^j + \mathrm{dg}b_{i-j} t^{i-j}).$$

If $i > k$ then the term b_0 does not occur in $\sum_{j=0}^{k} d_j b_{i-j}$ while it actually does if $0 \leq i \leq k$.

Therefore using (a) and (b) we get if $i > k$

$$\mathrm{dg}a_i + iR_1 < \mathrm{dg}d_k + kR_1 + \mathrm{dg}b_0$$

and if $0 \le i \le k$

$$dga_i + iR_1 \le dgd_k + kR_1 + dgb_0.$$

Since $d_k = 1$ and $a_0 = b_0\beta_1 \ldots \beta_k$ it follows that if $i > k$

$$dga_i + iR_1 < dga_0$$

and if $0 \le i \le k$

$$dga_i + iR_1 \le dga_0.$$

This means that $k = i_1$. □

Now we can prove theorem 2.2.

(i) Is obvious and (ii) follows from lemma 2.8.

(iii) We may again suppose that $h = 0$. We prove (iii) in case $k = 2$; the general case k follows then in the same way inductively from $k - 1$. We use the method of lemma 2.8.

For all t with $dgt \le R_2$ we have

$\forall N \in \mathbb{N}\ \exists i_0 \in \mathbb{N}$ such that for $i > i_0$

$$dga_i + idgt < -N.$$

Choose $n > \max(i_0, i_2)$ and define

$$P_N(t) = a_0 + a_1 t + \ldots + a_n t^n$$

and

$$g(t) = P_N(t) + g^*(t).$$

$P_N(t)$ has i_1 zeros $\beta_{N1}, \ldots, \beta_{Ni_1}$ such that $dg\beta_{Nj} = R_1$ $(1 \le j \le i_1)$ and $i_2 - i_1$ zeros $\beta_{N,i_1+1}, \ldots, \beta_{N,i_2}$ such that $dg\beta_{Nj} = R_2$ $(i_1+1 \le j \le i_2)$. Now we write

$$P_N(t) = a \prod_{j=1}^{i_1} (t-\beta_{Nj}) \prod_{j=i_1+1}^{i_2} (t-\beta_{Nj}) \prod_{\nu=i_2+1}^{n} (1 - \frac{t}{\beta_{N\nu}}),$$

where

$$a = \frac{(-1)^{i_2} a_0}{\prod_{j=1}^{i_2} \beta_{Nj}}.$$

Note that $dga = dga_0 - i_1 R_1 - (i_2 - i_1) R_2$, that a does not depend on N and that $dg\beta_{N\nu} > R_2$ for $\nu=i_2+1,\ldots,n$.

For $R_1 < dgt \le R_2$ we have

$$\sum_{j=i_1+1}^{i_2} dg(t-\beta_{Nj}) = dgP_N(t) - dga_0 + i_1R_1 + (i_2-i_1)R_2 - i_1R_2. \qquad (*)$$

Let $P_N(t)$ have ρ different zeros β with $dg\beta = R_2$ $(\rho\le i_2-i_1)$ and let $g(t)$ have the different zeros $t_1,\ldots,t_m$ with $R_1 < dgt_i \le R_2$. Define

$$\lambda = \min_{\substack{i\neq j\\ 1\le i,j\le m}} dg(t_i-t_j)$$

and the disjoint sets

$$U_{\lambda j} = \{t \mid dg(t-t_j) < \lambda\}, \qquad j = 1,\ldots,m.$$

From (*) it follows that for $i = 1,\ldots,m$ there exists at least one $j_i \in \{1,2,\ldots,\rho\}$ such that

$$dg(t_i-\beta_{Nj_i}) < \frac{-N-dga_0+i_1R_1+(i_2-2i_1)R_2}{i_2-i_1}.$$

Now choose N such that

$$\frac{-N-dga_0+i_1R_1+(i_2-2i_1)R_2}{i_2-i_1} < \max(\lambda,R_2)$$

then $\beta_{Nj_i} \in U_{\lambda i}$, $i = 1,\ldots,m$ and $dgt_i = dg\beta_{Nj_i} = R_2$.

From the box principle of DIRICHLET we conclude that $m \le \rho$, which proves that $g(t)$ has $m \le \rho \le i_2-i_1$ different zeros β with $dg\beta = R_2$.

Now we prove that $g(t)$ has exactly i_2-i_1 zeros β with $dg\beta = R_2$. Let $\beta_1,\ldots,\beta_{i_1}$ be the zeros of $g(t)$ with $dg\beta_i = R_1$ $(1\le i\le i_1)$ enumerated according to their multiplicities. Let $\beta_{i_1+1},\ldots,\beta_k$ be the zeros $t_1,\ldots,t_m$ of $g(t)$ enumerated according to their multiplicities where

$$k = \sum_{\nu=1}^{m} \text{multiplicity of } t_\nu.$$

Define

$$h(t) = \frac{g(t)}{\prod_{i=1}^{k} (t-\beta_i)}$$

$$Q(t) = \prod_{i=1}^{k} (t-\beta_i) = \sum_{\nu=0}^{k} d_\nu t^\nu$$

$$h(t) = \sum_{\nu=0}^{\infty} b_\nu t^\nu .$$

$h(t)$ has no zeros in $\{t \mid dgt \le R_2\}$ and therefore

(a) $$dgb_0 > dgb_i + iR_2 \qquad \text{for } i = 1,2,\ldots .$$

According to lemma 2.1 we have

$$k = \max_{i>i_1} \{i \mid \frac{dgd_{i_1} - dgd_i}{i-i_1} = R_2\}$$

and

$$i_1 = \max_{i>0} \{i \mid \frac{dgd_0 - dgd_i}{i} = R_1\};$$

hence we get for $i = i_1+1,\ldots,k$

$$dgd_i + iR_2 \le dgd_{i_1} + i_1R_2 = dgd_k + kR_2$$

and for $i = 0,1,\ldots,i_1$ we get

$$\begin{aligned} dgd_i + iR_2 &= dgd_i + iR_1 + i(R_2-R_1) = \\ &\le dgd_0 + i(R_2-R_1) = dgd_{i_1} + iR_2 + (i_1-i)R_1 \\ &< dgd_{i_1} + iR_2 + (i_1-i)R_2 = dgd_k + kR_2 . \end{aligned}$$

Now we have proved the relation

(b) $$dgd_i + iR_2 \le dgd_k + kR_2 \qquad \text{for } i = 0,1,\ldots,k.$$

Hence for all t with $dgt = R_2$ we have

$$dga_i + iR_2 \le \max_{0\le j\le k} (dgd_j t^j + dgb_{i-j} t^{i-j}).$$

For $i > k$ we get using (a) and (b) that

$$dga_i + iR_2 < dgd_k + kR_2 + dgb_0.$$

Since $d_k = 1$ and $a_0 = b_0\beta_1\cdots\beta_{i_1}\beta_{i_1+1}\cdots\beta_k$ this means

$$\begin{aligned} dga_i + iR_2 &< kR_2 + dga_0 - i_1R_1 - (k-i_1)R_2 \\ &= kR_2 + dga_{i_1} - (k-i_1)R_2 = dga_{i_1} + i_1R_2. \end{aligned}$$

Analogously for $i_1 \le i \le k$ we get using (b) that

$$dga_i + iR_2 \le dga_{i_1} + i_1R_2.$$

From the definition of i_2 it follows now that $k = i_2$, which proves our theorem. □

2.9. SPECIAL CASES

a) The function

$$\psi(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{q^k}}{F_k}$$

converges for all $t \in \Phi$ and $\psi(t)$ has a zero of order 1 in $t = 0$.

$$R_1 = \min_{i>1} \frac{0-dga_i}{i-1} = \min_{i>1} \left(\frac{q}{q-1}, \ldots, \frac{kq^k}{q^k-1}, \ldots\right) = \frac{q}{q-1}.$$

$$i_1 = \max_{i>1} \{i \mid \frac{-dga_i}{i-1} = \frac{q}{q-1}\} = q.$$

Hence $\psi(t)$ has $q-1$ zeros β with $dg\beta = \frac{q}{q-1}$, namely

$$\psi(c\xi) = 0$$

for $c \in \mathbb{F}_q\backslash\{0\}$ and

$$dgc\xi = dg\xi = \frac{q}{q-1} = 1 + \frac{1}{q-1}.$$

For $k \ge 2$ we have

$$R_k = \min_{i>q^{k-1}} \left(\frac{-(k-1)q^{k-1}-dga_i}{i-q^{k-1}}\right) = \frac{-(k-1)q^{k-1}+kq^k}{q^k-q^{k-1}} = k + \frac{1}{q-1}$$

and

$$i_k = q^k .$$

The function $\psi(t)$ has $q^k - q^{k-1}$ zeros β with $dg\beta = k + \frac{1}{q-1}$, namely $\psi(E\xi) = 0$ (see §1) and the number of different polynomials over $\mathbb{F}_q$ of degree k-1 is $q^k - q^{k-1}$.

b) In an analogous way it follows that the function

$$f(t) = \sum_{k=0}^{\infty} c_k \frac{t^{q^k}}{F_k} \quad \text{with } c_k \in \mathbb{F}_q,\ c_k \neq 0 \text{ for } k = 0,1,\dots$$

has:

a zero of order 1 in $t = 0$,
$q-1$ zeros β with $dg\beta = \frac{q}{q-1}$,
$\vdots$
q^k-q^{k-1} zeros β with $dg\beta = k + \frac{1}{q-1}$.

c) The function

$$J_n(t) = \sum_{r=0}^{\infty} (-1)^r \frac{t^{q^{n+r}}}{F_{n+r} F_r^{q^n}}, \quad n \in \mathbb{N} \cup \{0\}$$

has a zero of order q^n in $t = 0$ and $q^{n+k+1}-q^{n+k}$ zeros β with $dg\beta = n + 2k + \frac{2q}{q-1}$.

Remark: the function $J_n(t)$ was introduced by L. CARLITZ in 1960; see [2].

3. THE TRANSCENDENCY OF THE ZEROS OF $\psi(t)$, $f(t)$ AND $J_n(t)$

3.1. <u>THEOREM</u>. *Let* $\eta \neq 0$ *be a zero of the function*

$$f(t) = \sum_{j=0}^{\infty} c_j \frac{t^{q^j}}{F_j},$$

with $c_j \in \mathbb{F}_q$ *and* $c_j \neq 0$ *for* $j = 0,1,2,\dots$. *Then* η *is trancendental over* $\mathbb{F}_q\{x\}$.

For the proof of this theorem we use several lemmas.

3.2. LEMMA. *Let* n, $k_1,\ldots,k_n$, β *be non-negative integers. If*

$$q^{k_1} + \ldots + q^{k_n} \leq q^{\beta} + q^{\beta-1} + \ldots + q^{\beta-n+1}$$

and

$$k_1 \geq k_2 \geq \ldots \geq k_n$$

then

$$K' = \frac{F_\beta F_{\beta-1}\cdots F_{\beta-n+1}}{F_{k_1}F_{k_2}\cdots F_{k_n}}$$

is a polynomial.

PROOF. See [8], lemma 5.2. □

3.3. LEMMA. *A symmetric polynomial in the roots of a monic polynomial with coefficients in* $\mathbb{F}_q[x]$ *is itself an element of* $\mathbb{F}_q[x]$.

PROOF. For instance see [6]. □

3.4. LEMMA. *The equation*

$$q^{k_1} + q^{k_2} + \ldots + q^{k_i} = q^{\beta}$$

with β *a non-negative integer, has a solution* $k_1,k_2,\ldots,k_i \in \mathbb{Z}$ *with* $k_1 \geq k_2 \geq \ldots \geq k_i \geq 0$ *if and only if*

$$i = q^{\mu} + \lambda(q-1),$$

where

$$0 \leq \mu \leq \beta;\ \lambda \in \{0,1,\ldots,q^{\mu}-1\}.$$

If $\lambda = 0$ *then the only possible solution is* $k_1 = k_2 = \ldots = k_i = \beta - \mu$. *If* $\lambda > 0$ *then for every solution* $k_1 = k_2 = \ldots = k_{q^{\mu}-1} = \beta - \mu$.

PROOF. Suppose $i = q^{\mu} + \lambda(q-1)$ for a certain $\mu \in \{0,1,\ldots,\beta\}$ and $\lambda \in \{0,1,\ldots,\min(q^{\mu}-1,\beta-\mu)\}$ and suppose $k_1,\ldots,k_i \in \mathbb{Z}$ such that $k_1 \geq \ldots \geq k_i \geq 0$, satisfying

$$\begin{aligned}
&k_1 = k_2 = \dots = k_{q^\mu-2} = k_{q^\mu-1} && = \beta - \mu \\
&k_{q^\mu} = k_{q^\mu+1} = \dots = k_{q^\mu+q-2} && = \beta - \mu - 1 \\
&k_{q^\mu+q-1} = \dots = k_{q^\mu+2(q-1)-1} && = \beta - \mu - 2 \\
&\vdots \\
&k_{q^\mu+(\lambda-1)(q-1)} = \dots = k_{q^\mu+\lambda(q-1)-1} && = \beta - \mu - \lambda \\
&\qquad\qquad k_{q^\mu+\lambda(q-1)} && = \beta - \mu - \lambda
\end{aligned}$$

then

$$q^{k_1} + q^{k_2} + \dots + q^{k_i} = (q^\mu-1)q^{\beta-\mu} + (q-1)q^{\beta-\mu-1} + \dots$$
$$\dots + (q-1)q^{\beta-\mu-\lambda+1} + q.q^{\beta-\mu-\lambda} = q^\beta.$$

Now let i be an arbitrary natural number. Then there exist non-negative integers μ and $\lambda \in \{0,1,\dots,q^\mu-1\}$ such that

$$(1)\qquad q^\mu + \lambda(q-1) \le i < q^\mu + (\lambda+1)(q-1)$$

Furthermore we suppose that there exist a solution $k_1,\dots,k_i \in \mathbb{Z}$, $k_1 \ge \dots \ge k_i \ge 0$ of $q^{k_1}+\dots+q^{k_i} = q^\beta$. Then

$$q^\beta = q^{k_1}+\dots+q^{k_i} \le iq^{k_1} < q^{k_1}(q^\mu+(\lambda+1)(q-1)) < q^{k_1+\mu+1},$$

which gives

$$k_1 \ge \beta - \mu.$$

On the other hand it follows from $q^{k_1}+\dots+q^{k_i} = q^\beta$ that

$$k_1 \le \beta - 1.$$

Now write $k_1 = \beta - j$ for a certain $j \in \mathbb{Z}$ with $1 \le j \le \mu$. Then

$$q^{k_1}+\dots+q^{k_i} \le (q^j-1)q^{\beta-j} + (q-1)q^{\beta-j-1}+\dots+(q-1)q^{\beta-j-t} +$$
$$+ vq^{\beta-j-t-1},$$

where

$$q^j - 1 + t(q-1) + v = i,$$

$$0 < v \leq q,$$

$$\beta - j - t - 1 \geq 0.$$

If $0 < v < q$, then

$$q^{k_1} + \ldots + q^{k_i} \leq q^\beta - q^{\beta-j-t} + (q-1)q^{\beta-j-t-1} < q^\beta,$$

which contradicts $q^{k_1}+\ldots+q^{k_i} = q^\beta$, and therefore $v = q$. This means

(2) $$i = q^j + (t+1)(q-1).$$

Combining (1) and (2) we find

$$q^\mu + \lambda(q-1) \leq q^j + (t+1)(q-1) < q^\mu + (\lambda+1)(q-1)$$

and hence if $1 \leq j < \mu$:

$$\frac{q^\mu - q^j}{q-1} - 1 \leq t - \lambda < \frac{q^\mu - q^j}{q-1}$$

and if $j = \mu$:

$$\lambda \leq t + 1 < \lambda + 1.$$

Therefore if $j = \mu$ we find $t = \lambda - 1$. If $1 \leq j < \mu$ we remark that $\frac{q^\mu - q^j}{q-1}$ is a positive integer. Since t, λ are (non-negative) integers the only possible value for t is

$$t = \lambda - 1 + \frac{q^\mu - q^j}{q-1}$$

and therefore

$$i = q^j + (\lambda + \frac{q^\mu - q^j}{q-1})(q-1) = q^\mu + \lambda(q-1).$$

Hence a solution of $q^{k_1}+\ldots+q^{k_i} = q^\beta$ with $k_1 \geq \ldots \geq k_i \geq 0$ is only possible if

$$i = q^\mu + \lambda(q-1)$$

for some non-negative integer μ and $\lambda \in \{0,1,\ldots,q^\mu-1\}$. If $k_1 < \beta - \mu$, then

$$q^{k_1}+\ldots+q^{k_i} \le iq^{\beta-\mu-1} = (q^{\mu}+\lambda(q-1))q^{\beta-\mu-1} <$$
$$< (q^{\mu}+q^{\mu}(q-1))q^{\beta-\mu-1} = q^{\beta},$$

which contradicts $q^{k_1}+\ldots+q^{k_i} = q^{\beta}$. Hence $k_1 = \beta - \mu$. Since we want to have $k_1 \ge 0$ we must have $\mu \le \beta$. Hence a solution of

$$(3) \qquad q^{k_1} + q^{k_2} + \ldots + q^{k_i} = q^{\beta}$$

in integers $k_1,\ldots,k_i$ with $k_1 \ge k_2 \ge \ldots \ge k_i \ge 0$ is possible if and only if

$$(4) \qquad i = q^{\mu} + \lambda(q-1),$$

where $0 \le \mu \le \beta$ and $0 \le \lambda \le q^{\mu}-1$ are integers.

Let $i_0 \ge 1$ be the smallest index such that

$$k_1 = k_2 = \ldots = k_{i_0} = \beta - \mu$$

and

$$k_{i_0+1} \le \beta - \mu - 1.$$

Then

$$q^{k_1} + \ldots + q^{k_i} = i_0 q^{\beta-\mu} + q^{k_{i_0+1}} + \ldots + q^{k_i}$$

and

$$(5) \qquad i_0 q^{\beta-\mu} + (i-i_0) \le q^{\beta} \le i_0 q^{\beta-\mu} + (i-i_0)q^{\beta-\mu-1}.$$

From the left inequality of (5) it follows that

$$i_0(q^{\beta-\mu}-1) \le q^{\beta} - q^{\mu}$$

and therefore

$$i_0 \le q^{\mu}.$$

If $i_0 < q^{\mu} - 1$ then $1 \le i_0 \le q^{\mu} - 2$ and then it follows from the right inequality of (5) that

$$q^\beta \le (q^\mu-2)q^{\beta-\mu} + (q^\mu+\lambda(q-1)-1)q^{\beta-\mu-1} \le$$

$$\le q^\beta - 2q^{\beta-\mu} + (q^\mu-1)q^{\beta-\mu} < q^\beta + q^\beta(1-\frac{2}{q^\mu}).$$

Since $1 \le i_0 \le q^\mu - 2$ it follows that $\mu \ge 1$ and hence $q^\mu \ge q \ge 2$ which means

$$q^\beta < q^\beta + q^\beta(1-\frac{2}{q^\mu}) \le q^\beta.$$

Hence from (5) it follows that

$$q^\mu - 1 \le i_0 \le q^\mu.$$

If $\lambda = 0$ then suppose $i_0 = q^\mu - 1$, then

$$q^\beta = q^{k_1}+\ldots+q^{k_i} \le (q^\mu-1)q^{\beta-\mu} + (i-i_0)q^{\beta-\mu-1} =$$

$$= q^\beta - q^{\beta-\mu} + q^{\beta-\mu-1} < q^\beta;$$

contradiction. Hence, if $\lambda = 0$ then $i_0 = q^\mu$.

If $1 \le \lambda \le q^\mu - 1$ then suppose $i_0 = q^\mu$, then

$$q^\beta = q^{k_1}+\ldots+q^{k_i} \ge q^\mu(q^{\beta-\mu}) + (i-i_0).1 =$$

$$= q^\beta + (q^\mu+\lambda(q-1)-q^\mu) \ge q^\beta + q - 1 > q^\beta;$$

contradiction. Hence, if $\lambda > 0$, then $i_0 = q^\mu - 1$.

This proves our lemma. □

3.5. LEMMA. *The equation*

(1) $$q^{k_1} + q^{k_2} + \ldots + q^{k_i} = q^\beta$$

with β *a non-negative integer has a solution* $k_1,k_2,\ldots,k_i \in \mathbb{Z}$ *with* $k_1 \ge k_2 \ge \ldots \ge k_i \ge 0$ *if and only if*

$$i = q^{\mu_0} + q^{\mu_1} + \ldots + q^{\mu_r} - r$$

where $\beta \ge \mu_0 \ge \mu_1 \ge \ldots \ge \mu_r \ge 1$ *and* $\mu_0 + \mu_1 + \ldots + \mu_r \le \beta$ *if* $r > 0$ *and* $\beta \ge \mu_0 \ge 0$ *if* $r = 0$.

For this i *the only possible solution is given by*

$$\begin{cases} k_1 = k_2 = \dots = k_{q^{\mu_0}-1} & = \beta - \mu_0 \\ k_{q^{\mu_0}} = \dots\dots = k_{q^{\mu_0}+q^{\mu_1}-2} & = \beta - \mu_0 - \mu_1 \\ \vdots & \vdots \\ k_{q^{\mu_0}+q^{\mu_1}+\dots+q^{\mu_{r-1}}-r+1} = \dots = k_{q^{\mu_0}+q^{\mu_1}+\dots+q^{\mu_r}-(r+1)} & = \beta - \mu_0 - \mu_1 - \dots - \mu_r \\ \qquad\qquad k_{q^{\mu_0}+q^{\mu_1}+\dots+q^{\mu_r}-r} & = \beta - \mu_0 - \mu_1 - \dots - \mu_r. \end{cases}$$

PROOF. According to lemma 3.4 the equation (1) is solvable if and only if

$$i = q^{\mu} + \lambda(q-1)$$

for a $\mu \in [0,\beta]$ and $0 \le \lambda \le q^{\mu} - 1$.

If $\lambda = 0$ the lemma is proved with $\mu_0 = \mu$ and $r = 0$ by lemma 3.4.

Now suppose $\lambda \neq 0$. Define

$$\mu_0 = \mu.$$

Then from lemma 3.4 it follows that

$$q^{k_1} + \dots + q^{k_i} = q^{\beta}$$

where

$$k_1 = \dots = k_{q^{\mu_0}-1} = \beta - \mu_0.$$

Hence

$$q^{k_{q^{\mu_0}}} + \dots + q^{k_i} = q^{\beta-\mu_0}. \tag{2}$$

According to lemma 3.4 equation (2) is solvable if and only if

$$i - q^{\mu_0} + 1 = q^{\mu_1} + \lambda_1(q-1)$$

for a $\mu_1 \in \{0,1,\dots,\beta-\mu_0\}$ and $\lambda_1 \in \{0,1,\dots,q^{\mu_1}-1\}$, which means since $i = q^{\mu_0} + \lambda(q-1)$ that

$$\lambda = \frac{q^{\mu_1}-1}{q-1} + \lambda_1.$$

If $\mu_1 = 0$, then $\lambda = \lambda_1 \leq q^{\mu_1} - 1 = 0$, in contradiction with our assumption that $\lambda \neq 0$. Hence $\mu_1 \geq 1$.

If $\lambda_1 = 0$ then the process is ended and

$$i = q^{\mu_0} + q^{\mu_1} - 1,$$

where $1 \leq \mu_1$ and since μ_1 is a non-negative integer for which

$$q^{\mu_1} = 1 + \lambda(q-1) < 1 + q^{\mu_0}(q-1) \leq q^{\mu_0+1}$$

it follows that $\mu_1 \leq \mu_0$. Furthermore

$$k_1 = \ldots = k_{q^{\mu_0}-1} = \beta - \mu_0$$

and

$$k_{q^{\mu_0}} = \ldots = k_{q^{\mu_0}+q^{\mu_1}-1} = \beta - \mu_0 - \mu_1.$$

If $\lambda_1 \neq 0$ we proceed in an analogous way.

Since $\mu_{j+1} \leq \beta - \mu_0 - \mu_1 - \ldots - \mu_j$ and $0 \leq \lambda_{j+1} \leq q^{\mu_{j+1}} - 1$ this process must end after a finite number of steps. □

3.6. LEMMA. *Let* $k_1 \geq \ldots \geq k_n \geq 0$ *be integers such that*

$$q^{\nu} < q^{k_1} + q^{k_2} + \ldots + q^{k_n} < q^{\nu+1},$$

then there exists an integer i *with* $1 \leq i < n$ *such that*

$$q^{k_1} + q^{k_2} + \ldots + q^{k_i} = q^{\nu}.$$

PROOF. See WADE [8], lemma 5.10. □

Now we are able to prove theorem 3.1.

PROOF.

$$f(t) = \sum_{j=0}^{\infty} c_j \frac{t^{q^j}}{F_j}$$

with $c_j \in \mathbb{F}_q$, $c_j \neq 0$ for $j = 0,1,2,\dots$.

Let $\eta \neq 0$ be a zero of $f(t)$. According to Corollary 2.9

$$dg\eta = \frac{q}{q-1} + \lambda,$$

where λ is some non-negative integer.

Suppose η is algebraic. Then there exists an $e \in \mathbb{N}$ such that $\eta^{q^e} = \alpha$ is separable. Let m be the degree of the minimal-polynomial for α.

$$P(t) = t^m + \frac{A_{m-1}}{B_{m-1}} t^{m-1} + \dots + \frac{A_0}{B_0} \text{ with } A_i, B_i \in \mathbb{F}_q[x].$$

$$P(\alpha) = 0.$$

Since α is separable, $dg\alpha = \dfrac{dgA_0 - dgB_0}{m}$, which will be denoted by $\dfrac{a_0}{m}$. Hence $a_0 = mq^e(\frac{q}{q-1} + \lambda)$. Let $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_m$ be the algebraic conjugated elements of α; then $dg\alpha_i = \frac{a_0}{m}$ for $i = 1,\dots,m$. There exists a unique $s \in \mathbb{N}$ such that

$$q^{s-1} < m \leq q^s.$$

Now multiply $P(t)$ with the factor $(t - x^{\lambda q^e})^{q^s - m}$, then

$$Q(t) = P(t)(t - x^{\lambda q^e})^{q^s - m}$$

is a polynomial over $\mathbb{F}_q\{x\}$ of degree $n = q^s$ whose roots are

$$\alpha_1, \dots, \alpha_m;\ \alpha_{m+1} = \dots = \alpha_n = x^{\lambda q^e}.$$

Furthermore

$$Q(t) = t^n + \frac{E_{n-1}}{D_{n-1}} t^{n-1} + \dots + \frac{E_0}{D_0},$$

where

$$dg\frac{E_0}{D_0} = dg(\alpha_1 \cdot \ldots \cdot \alpha_n) = dg(\alpha_1 \cdot \ldots \cdot \alpha_m) + (n-m)\lambda q^e =$$

$$= mq^e(\frac{q}{q-1} + \lambda) + (n-m)\lambda q^e = mq^e \cdot \frac{q}{q-1} + n\lambda q^e.$$

Note that $dg(\alpha_1 \cdot \ldots \cdot \alpha_n) = mq^e \cdot \frac{q}{q-1} + n\lambda q^e$.

Let β be a natural number, which will be fixed later. Denote by

$$D := D_0 D_1 \ldots D_{n-1}$$

and by

$$K_\beta := F_\beta F_{\beta-1} \ldots F_{\beta-n+1}.$$

Remark that $D\alpha_i$ is an algebraic integer.

Since $f(\eta) = 0$ we have

$$0 = f^{q^e}(\eta) = \left(\sum_{k=0}^{\infty} c_k \frac{\eta^{q^k}}{F_k}\right)^{q^e} = \sum_{k=0}^{\infty} c_k \frac{\eta^{q^{k+e}}}{F_k^{q^e}} = \sum_{k=0}^{\infty} c_k \frac{\alpha^{q^k}}{F_k^{q^e}},$$

and therefore

$$0 = \prod_{i=1}^{n} \left(\sum_{k=0}^{\infty} \frac{c_k}{F_k^{q^e}} \alpha_i^{q^k}\right) =$$

$$= \sum_{k_1 \geq \ldots \geq k_n \geq 0} \frac{c_{k_1} \ldots c_{k_n}}{F_{k_1}^{q^e} \ldots F_{k_n}^{q^e}} \sum_{(i_1, \ldots, i_n)} \alpha_{i_1}^{q^{k_1}} \alpha_{i_2}^{q^{k_2}} \ldots \alpha_{i_n}^{q^{k_n}},$$

where the sum is taken over all terms $\alpha_{i_1}^{q^{k_1}} \alpha_{i_2}^{q^{k_2}} \ldots \alpha_{i_n}^{q^{k_n}}$ which are different. Now the following product

$$0 = K_\beta^{q^e} D^{q^{\beta+1}} \prod_{i=1}^{n} \left(\sum_{k=0}^{\infty} \frac{c_k}{F_k^{q^e}} \alpha_i^{q^k}\right)$$

can be written as

$$K_\beta^{q^e} D^{q^{\beta+1}} \sum_{\nu=s}^{\infty} \sum_{\substack{k_1 \geq \ldots \geq k_n \geq 0 \\ q^\nu \leq q^{k_1} + \ldots + q^{k_n} < q^{\nu+1}}} \frac{c_{k_1} \ldots c_{k_n}}{F_{k_1}^{q^e} \ldots F_{k_n}^{q^e}} \sum_{(i_1, \ldots, i_n)} \alpha_{i_1}^{q^{k_1}} \alpha_{i_2}^{q^{k_2}} \ldots \alpha_{i_n}^{q^{k_n}} = 0.$$

We split this sum into two parts

$$I + Q = 0,$$

where I is the sum over the terms with $\nu = s, \ldots, \beta$ and Q is the sum over the other terms. Since

$$\sum_{(i_1,\dots,i_n)} (D\alpha_{i_1})^{q^{k_1}} (D\alpha_{i_2})^{q^{k_2}} \dots (D\alpha_{i_n})^{q^{k_n}}$$

is a symmetric polynomial in the roots of a monic polynomial with coefficients in $\mathbb{F}_q[x]$ the sum itself is an element of $\mathbb{F}_q[x]$ according to lemma 3.3. Furthermore from lemma 3.2 it follows that

$$\frac{K_\beta^{q^e}}{F_{k_1}^{q^e}\dots F_{k_n}^{q^e}} = \left(\frac{F_\beta F_{\beta-1}\dots F_{\beta-n+1}}{F_{k_1}F_{k_2}\dots F_{k_n}}\right)^{q^e}$$

is an element of $\mathbb{F}_q[x]$ for all terms of I, since the maximal term $q^{k_1}+\dots+q^{k_n}$ with $k_1 \geq k_2 \geq \dots \geq k_n$ such that $q^{k_1}+\dots+q^{k_n} < q^{\beta+1}$ is the term with $k_1 = \beta$, $k_2 = \beta - 1,\dots,k_n = \beta - n + 1$. Hence $I \in \mathbb{F}_q[x]$, which means either $\mathrm{dg}I \geq 0$ or $I = 0$.

Our aim is to show that for β chosen large enough $\mathrm{dg}Q < 0$. Then it follows from $I + Q = 0$ that $I = 0$ and $Q = 0$. Furthermore we shall prove that $\mathrm{dg}Q > -\infty$, hence $Q \neq 0$. This will give the desired contradiction.

The proof that $\mathrm{dg}Q < 0$ will be split in several parts. Every term of Q has the form

$$K_\beta^{q^e} D^{q^{\beta+1}} \frac{c_{k_1}\dots c_{k_n}}{F_{k_1}^{q^e}\dots F_{k_n}^{q^e}} \sum_{(i_1,\dots,i_n)} \alpha_{i_1}^{q^{k_1}}\dots\alpha_{i_n}^{q^{k_n}}, \tag{1}$$

with $k_1 \geq \dots \geq k_n \geq 0$, $q^\nu \leq q^{k_1}+\dots+q^{k_n} < q^{\nu+1}$ and $\nu \geq \beta + 1$.

According to lemma 3.4, since $n = q^s$, there exists exactly one solution $k_1 \geq \dots \geq k_n \geq 0$ of $q^{k_1}+\dots+q^{k_n} = q^\nu$ and this solution is given by $k_1 = \dots = k_n = \nu - s$. Let N_ν denote the term of Q where

$$q^{k_1}+\dots+q^{k_n} = q^\nu \qquad (\nu = \beta+1,\dots)$$

We shall prove

(i) If β is chosen sufficiently large then

$$\mathrm{dg}N_{\nu+j} - \mathrm{dg}N_\nu < 0 \qquad \text{for } j = 1,2,\dots\ .$$

Hence for β chosen sufficiently large we have

$$dgN_\nu - dgN_{\beta+1} < 0 \qquad \text{for } \nu = \beta+2, \ldots .$$

(ii) Let N be a term of Q with

$$q^\nu < q^{k_1} + \ldots + q^{k_n} < q^{\nu+1} \qquad (\nu = \beta+1, \ldots),$$

then

$$dgN \dot{-} dgN_\nu < 0$$

if β is chosen large enough. And for all $N \neq N_{\beta+1}$ in Q we have

$$dgN - dgN_{\beta+1} < 0.$$

Proof of (i): Since $k_1 = \ldots = k_n = \nu - s$ we have

$$N_\nu = K_\beta^{q^e} D^{q^{\beta+1}} \left(\frac{c_{\nu-s}}{F_{\nu-s}^{q^e}}\right)^n \left(\sum_{(i_1,\ldots,i_n)} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_n}\right)^{q^{\nu-s}}$$

$$= K_\beta^{q^e} D^{q^{\beta+1}} \left(\frac{c_{\nu-s}}{F_{\nu-s}^{q^e}}\right)^n (\alpha_1\ldots\alpha_n)^{q^{\nu-s}} .$$

Hence since $c_{\nu-s} \neq 0$ we have $N_\nu \neq 0$ and

$$dgN_\nu = q^e dgK_\beta + q^{\beta+1}dgD - nq^e(\nu-s)q^{\nu-s} + q^{\nu-s}dg(\alpha_1\ldots\alpha_n),$$

which yields

$$\text{(2)} \qquad dgN_\nu = q^e dgK_\beta + q^{\beta+1}dgD - (\nu-s)q^{\nu+e} + q^{\nu+e}\left(\frac{m}{n} \cdot \frac{q}{q-1} + \lambda\right)$$

For $\nu = \beta + 1$ we get

$$\begin{aligned} dgN_{\beta+1} &= q^e(\beta q^\beta + \ldots + (\beta-n+1)q^{\beta-n+1}) + q^{\beta+1}dgD - (\beta+1-s)q^{\beta+e+1} + \\ &\quad + q^{\beta+1+e}\left(\frac{m}{n} \cdot \frac{q}{q-1} + \lambda\right) \\ &\leq \beta q^e(q^\beta + \ldots + q^{\beta-n+1}) + q^{\beta+1+e}\left(-\beta - 1 + s + \frac{dgD}{q^e} + \frac{m}{n} \cdot \frac{q}{q-1} + \lambda\right) \\ &= q^{\beta+e-n+1}\left[\beta(q^{n-1} + \ldots + q + 1) + q^n\left(-\beta - 1 + s + \frac{dgD}{q^e} + \frac{m}{n} \cdot \frac{q}{q-1} + \lambda\right)\right]. \end{aligned}$$

Since

$$-q^n + q^{n-1} + \ldots + q + 1 < 0$$

and since

$$q^n(-1 + s + \frac{dgD}{q^e} + \frac{m}{n} \cdot \frac{q}{q-1} + \lambda)$$

is a constant which does not depend on β, there exists a $\beta_1 \in \mathbb{N}$ such that

(3) $\qquad dgN_{\beta+1} < 0 \quad$ for all $\beta > \beta_1$.

Furthermore we have for $j \geq 1$

$$dgN_{\nu+j} - dgN_\nu = -(\nu+j-s)q^{\nu+j+e} + q^{\nu+j+e}(\frac{m}{n} \cdot \frac{q}{q-1} + \lambda) +$$
$$+ (\nu-s)q^{\nu+e} - q^{\nu+e}(\frac{m}{n} \cdot \frac{q}{q-1} + \lambda) =$$
$$= -jq^{\nu+j+e} + (\nu - s - \frac{m}{n} \cdot \frac{q}{q-1} - \lambda)(q^{\nu+e} - q^{\nu+j+e}).$$

Since

$$\nu - s - \frac{m}{n} \cdot \frac{q}{q-1} - \lambda \geq \beta + 1 - s - \frac{m}{m} \cdot \frac{q}{q-1} - \lambda > 0 \quad \text{for } \beta > \beta_2$$

we have

(4) $\qquad dgN_{\nu+j} - dgN_\nu < 0 \quad$ for $j \geq 1$ and $\beta > \beta_2$.

Hence from (3) and (4) we conclude (i).

<u>Proof</u> of (ii): N is a term such that $q^\nu < q^{k_1} + \ldots + q^{k_n} < q^{\nu+1}$. According to lemma 3.6 there exists an $i \in \{1,2,\ldots,n-1\}$ such that

$$q^{k_1} + \ldots + q^{k_i} = q^\nu.$$

According to lemma 3.5 this is only possible if

$$i = q^\mu \quad \text{and then} \quad k_1 = \ldots = k_i = \nu - \mu$$

or

$$i = q^{\mu_0} + q^{\mu_1} + \ldots + q^{\mu_r} - r$$

with

$$\mu_0 \geq \mu_1 \geq \ldots \geq \mu_r \geq 1 \quad \text{and} \quad \mu_0 + \mu_1 + \ldots + \mu_r \leq \nu .$$

(iia) First we consider the case that

$$i = q^{\mu} \quad \text{and} \quad k_1 = \ldots = k_i = \nu - \mu .$$

Then

$$N = K_\beta^{q^e} D^{q^{\beta+1}} \left(\frac{c_{\nu-\mu}}{F_{\nu-\mu}^{q^e}} \right)^i \frac{c_{k_{i+1}} \cdots c_{k_n}}{F_{k_{i+1}}^{q^e} \cdots F_{k_n}^{q^e}} \sum_{(j_1,\ldots,j_n)} (\alpha_{j_1} \cdots \alpha_{j_i})^{q^{\nu-\mu}} \alpha_{j_{i+1}}^{q^{k_{i+1}}} \cdots \alpha_{j_n}^{q^{k_n}} ,$$

and therefore

$$\begin{aligned} \operatorname{dg} N \leq\ & q^e \operatorname{dg} K_\beta + q^{\beta+1} \operatorname{dg} D - i q^e (\nu-\mu) q^{\nu-\mu} + \\ & - k_{i+1} q^{k_{i+1}+e} - \ldots - k_n q^{k_n+e} + q^{\nu-\mu} . i . q^e \left(\frac{q}{q-1} + \lambda\right) + \\ & + \left(q^{k_{i+1}} + \ldots + q^{k_n}\right) . q^e \left(\frac{q}{q-1} + \lambda\right) . \end{aligned}$$

Since $N_\nu \neq 0$ using (2) we get

$$\begin{aligned} \operatorname{dg} N - \operatorname{dg} N_\nu \leq\ & -(\nu-\mu) q^{\nu+e} + q^{k_{i+1}+e} \left(\frac{q}{q-1} + \lambda - k_{i+1}\right) + \ldots \\ & \ldots + q^{k_n+e} \left(\frac{q}{q-1} + \lambda - k_n\right) + q^{\nu+e} \left(\frac{q}{q-1} + \lambda\right) + (\nu-s) q^{\nu+e} + \\ & - q^{\nu+e} \left(\frac{m}{n} . \frac{q}{q-1} + \lambda\right) \leqq (\mu-s) q^{\nu+e} + q^{\nu+e} . \frac{q}{q-1} \left(1 - \frac{m}{n}\right) + \\ & + (n-i) \max_{i+1 \leq j \leq n} q^{k_j+e} \left(\frac{q}{q-1} + \lambda - k_j\right) . \end{aligned}$$

Consider the function $g\colon \mathbb{R}^+ \to \mathbb{R}$ defined by

$$g(x) = q^{x+e} \left(\frac{q}{q-1} + \lambda - x\right) .$$

Then for $x \geq -\frac{1}{\ln q} + \frac{q}{q-1} + \lambda$ the function $g(x)$ is monotonically decreasing, hence

$$\max_{i+1\le j\le n} q^{k_j+e}\left(\frac{q}{q-1}+\lambda-k_j\right) \leqq \frac{1}{\ln q}\cdot q^{\frac{q}{q-1}+\lambda-\frac{1}{\ln q}+e}\ .$$

The constant $a_1 = \frac{1}{\ln q}\cdot q^{\frac{q}{q-1}+\lambda-\frac{1}{\ln q}+e} > 0$ does not depend on the choice of $k_{j+1},\dots,k_n$. Furthermore, since $q^{s-1} < m \le q^s = n$ we have

$$\frac{q}{q-1}\left(1-\frac{m}{n}\right) \le \frac{q}{q-1}\left(1-\frac{1}{n}-\frac{1}{q}\right) = 1-\frac{q}{n(q-1)} < 1.$$

Now we have

$$dgN - dgN_\nu \leqq (\mu-s)q^{\nu+e} + q^{\nu+e}\left(1-\frac{q}{n(q-1)}\right) + na_1.$$

Since $i \in \{1,\dots,n-1\}$ and $i = q^\mu$ and $n = q^s$ we have $\mu - s \le -1$. Therefore we have

$$dgN - dgN_\nu \leqq -q^{\nu+e} + q^{\nu+e}\left(1-\frac{q}{n(q-1)}\right) + na_1 < 0$$

for $\beta > \beta_3$. Using (i) we get for β sufficiently large

(5) $$dgN - dgN_{\beta+1} < 0.$$

(iib) Now we consider the case that N is a term such that $q^\nu < q^{k_1}+\dots+q^{k_n} < q^{\nu+1}$ and $q^{k_1}+\dots+q^{k_i} = q^\nu$ where $i = q^{\mu_0}+q^{\mu_1}+\dots+q^{\mu_r} - r$ with $\mu_0 \ge \mu_1 \ge \dots \ge \mu_r \ge 1$ and $\mu_0+\dots+\mu_r \le \nu$. Then according to lemma 3.5 we have

$$N = K_\beta^{q^e} D^{q^{\beta+1}} \sum_{k_1\ge\dots\ge k_n\ge 0} \frac{c_{k_1}\cdots c_{k_n}}{F_{k_1}^{q^e}\cdots F_{k_n}^{q^e}} \sum_{(j_1,\dots,j_n)} \alpha_{j_1}^{q^{k_1}}\alpha_{j_2}^{q^{k_2}}\cdots\alpha_{j_n}^{q^{k_n}},$$

where

$$\begin{cases} k_1 = \dots = k_{q^{\mu_0}-1} & = \nu - \mu_0 \\ k_{q^{\mu_0}} = \dots = k_{q^{\mu_0}+q^{\mu_1}-2} & = \nu - \mu_0 - \mu_1 \\ \vdots & \vdots \\ k_{q^{\mu_0}+q^{\mu_1}+\dots+q^{\mu_{r-1}}-r+1} = \dots = k_{q^{\mu_0}+q^{\mu_1}+\dots+q^{\mu_r}-(r+1)} & = \nu - \mu_0 - \dots - \mu_r \\ k_{q^{\mu_0}+q^{\mu_1}+\dots+q^{\mu_r}-r} & = \nu - \mu_0 - \mu_1 - \dots - \mu_r \end{cases}$$

and therefore:

$$\begin{aligned} dgN \le\ & q^e dgK_\beta + q^{\beta+1} dgD - (\nu-\mu_0) q^{\nu-\mu_0+e} (q^{\mu_0}-1) + \\ & -(\nu-\mu_0-\mu_1) q^{\nu-\mu_0-\mu_1+e} (q^{\mu_1}-1) - \dots - (\nu-\mu_0-\dots-\mu_r) q^{\nu-\mu_0-\dots-\mu_r+e} (q^{\mu_r}-1) + \\ & -(\nu-\mu_0-\dots-\mu_r) q^{\nu-\mu_0-\dots-\mu_r+e} + \\ + \{ & (q^{\mu_0}-1) q^{\nu-\mu_0} + (q^{\mu_1}-1) q^{\nu-\mu_0-\mu_1} + \dots + (q^{\mu_r}-1) q^{\nu-\mu_0-\dots-\mu_r} + q^{\nu-\mu_0-\dots-\mu_r} \} \max_j dg\alpha_j + \\ & -q^e (k_{i+1} q^{k_{i+1}} + \dots + k_n q^{k_n}) + (q^{k_{i+1}} + \dots + q^{k_n}) \max_j dg\alpha_j = \\ =\ & q^e dgK_\beta + q^{\beta+1} dgD - (\nu-\mu_0) q^{\nu+e} + q^\nu \max_j dg\alpha_j + \\ & + q^{k_{i+1}} (\max_j dg\alpha_j - k_{i+1} q^e) + \dots + q^{k_n} (\max_j dg\alpha_j - k_n q^e) \\ \le\ & q^e dgK_\beta + q^{\beta+1} dgD - (\nu-\mu_0) q^{\nu+e} + q^{\nu+e} \left(\frac{q}{q-1} + \lambda\right) + \\ & (n-i) q^e \max_{i+1 \le j \le n} \left(\frac{q}{q-1} + \lambda - k_j\right) q^{k_j} . \end{aligned}$$

Since $N_\nu \ne 0$ we have

$$\begin{aligned} dgN - dgN_\nu \le\ & (\mu_0 - s) q^{\nu+e} + q^{\nu+e} \frac{q}{q-1} \left(1 - \frac{m}{n}\right) + \\ & + (n-i) q^e \max_{i+1 \le j \le n} \left(\frac{q}{q-1} + \lambda - k_j\right) q^{k_j} . \end{aligned}$$

Using (iia) we get

$$dgN - dgN_\nu \le (\mu_0 - s)q^{\nu+e} + q^{\nu+e}\left(1 - \frac{q}{n(q-1)}\right) + na_1.$$

Since $\mu_1 \ge \mu_2 \ge \dots \ge \mu_r \ge 1$ we have

$$q^{\mu_0} = i + r - (q^{\mu_1} + q^{\mu_2} + \dots + q^{\mu_r}) \le i \le n - 1 = q^s - 1$$

and therefore $\mu_0 \le s - 1$. Hence we conclude for this N:

(6) $\qquad dgN - dgN_{\beta+1} < 0 \qquad$ for β sufficiently large.

Combining (5) and (6) we have proved (ii).

Now choose $\beta > \max(\beta_1, \beta_2, \beta_3)$ then for all terms N of Q we have

$$dgN < dgN_{\beta+1} \qquad \text{if } N \ne N_{\beta+1}.$$

Since $dgN_{\beta+1} < 0$ we conclude $I = 0$ and $Q = 0$. But since

$$dgQ = dgN_{\beta+1} = q^{\beta+e-n+1}[-\beta q^n + \beta q^{n-1} + (\beta-1)q^{n-2} + \dots + (\beta-n+1) + \\ + q^n(-1 + s + \frac{dgD}{q^e} + \frac{m}{n} \cdot \frac{q}{q-1} + \lambda)]$$

we have $Q \ne 0$. Hence we have the desired contradiction which proves the transcendency of η. $\square$

3.7. THEOREM. *Let $\eta \ne 0$ be a zero of the function*

$$J_n(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{q^{n+k}}}{F_k^{q^n} F_{n+k}} \qquad (n \in \mathbb{N}),$$

then η is transcendental over $\mathbb{F}_q\{x\}$.

PROOF. We follow the proof of theorem 3.1. According to Corollary 2.9

$$dg\eta = n + \frac{2q}{q-1} + \lambda$$

for some $\lambda \in \mathbb{N} \cup \{0\}$.

Suppose η is algebraic and η^{q^e} is separable. Let m be the degree of the minimalpolynomial P for $\alpha = \eta^{q^e}$. Let $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_m$ be the conjugated elements of α. There exists a unique s such that

$$q^{s-1} < m \le q^s.$$

Let

$$Q(t) := P(t)(t - x^{q^e(n+\lambda+2)})^{q^s-m}$$

then $Q(t)$ is a polynomial over $\mathbb{F}_q\{x\}$ of degree $N = q^s$, with roots $\alpha_1,\ldots,\alpha_m;\ \alpha_{m+1} = \ldots = \alpha_N = x^{q^e(n+\lambda+2)}$. The natural number β will be chosen later. Let D be a denominator for the coefficients of Q, then $D\alpha_i$ is an algebraic integer. Denote by

$$K_\beta := F_\beta F_{\beta-1}\ldots F_{\beta-N+1}.$$

Now

$$0 = J_n^{q^e}(\eta) = \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{q^{k+n}}}{F_k^{q^{n+e}} F_{n+k}^{q^e}}$$

and therefore

$$0 = D^{q^{\beta+n+1}} K_\beta^{q^{n+e}} K_{\beta+n}^{q^e} \sum_{\nu=s}^{\infty} \sum_{\substack{k_1\ge\ldots\ge k_N\ge 0\\ q^\nu\le q^{k_1}+\ldots+q^{k_N}<q^{\nu+1}}} \frac{(-1)^{k_1+\ldots+k_N} \sum_{(i_1,\ldots,i_N)} \alpha_{i_1}^{q^{k_1}}\ldots\alpha_{i_N}^{q^{k_N}}}{F_{k_1}^{q^{n+e}}\ldots F_{k_N}^{q^{n+e}} F_{k_1+n}^{q^e}\ldots F_{k_N+n}^{q^e}}.$$

We split this sum into two parts: $I + Q = 0$, where I denotes the sum over the terms with $\nu = s,\ldots,\beta$. Then I is a polynomial.

Let N_ν be the term of Q with $q^{k_1}+\ldots+q^{k_N} = q^\nu$, hence $k_1 = \ldots = k_N = \nu - s$. Then

$$N_\nu = D^{q^{\beta+1}} K_\beta^{q^{n+e}} K_{\beta+n}^{q^e} \left(\frac{(-1)^{\nu-s}}{F_{\nu-s}^{q^{n+e}} F_{n+\nu-s}^{q^e}}\right)^N (\alpha_1\ldots\alpha_N)^{q^{\nu+n-s}}$$

and

$$\text{(1)}\qquad dgN_\nu = q^{\beta+1}dgD + q^{n+e}dgK_\beta + q^e dgK_{\beta+n} + -q^{n+\nu+e}[2(\nu-s) - \frac{m}{N}\cdot\frac{2}{q-1} - \lambda - 2].$$

Especially:

$$dgN_{\beta+1} = q^{\beta+n+e+1-N}[-2\beta q^N+(2\beta+n)q^{N-1}+(2\beta+n-2)q^{N-2}+\ldots$$

$$\ldots+(2\beta+n-2N+2)+q^N(\frac{dgD}{q^e} + 2s + \frac{m}{N} \cdot \frac{2}{q-1} + \lambda)]$$

$$\leq q^{\beta+n+e+1-N}[2\beta(-q^N+q^{N-1}+\ldots+q+1)+q^N(\frac{dgD}{q^e} + 2s + \frac{m}{N} \cdot \frac{2}{q-1} + \lambda)].$$

Hence for β sufficiently large

(2) $$dgN_{\beta+1} < 0.$$

Furthermore for β large enough and all $\nu \geq \beta + 1$ we have

(3) $$dgN_{\nu+1} - dgN_\nu < (q^{n+\nu+e}-q^{n+\nu+1+e})[2(\nu-s-1) - \frac{m}{N} \cdot \frac{2}{q-1} - \lambda] < 0.$$

If N is a term such that $q^\nu < q^{k_1}+\ldots+q^{k_N} < q^{\nu+1}$ and $i = q^\mu$, $k_1 = \ldots = k_i = \nu - \mu$ and $q^\nu = q^{k_1}+\ldots+q^{k_i}$, then using (1) we get

$$dgN - dgN_\nu \leq 2q^{n+\nu+e}\{(\mu-s) + \frac{1}{q-1}(1 - \frac{m}{N})\} +$$

$$+ n \max_{i+1\leq j\leq N} q^{k_j+n+e}(\frac{2q}{q-1} + \lambda - 2k_j).$$

This maximum is less that a constant which does not depend on ν, hence for β sufficiently large

(4) $$dgN - dgN_\nu < 0.$$

If N is a term such that $q^\nu < q^{k_1}+\ldots+q^{k_N} < q^{\nu+1}$ and $i = q^{\mu_0} + q^{\mu_1}+\ldots+q^{\mu_r} - r$ with $\mu_0 \geq \mu_1 \geq \ldots \geq \mu_r \geq 1$ then (using lemma 3.5) we get

(5) $$dgN - dgN_\nu \leq 2q^{n+\nu+e}\{(\mu_0-s) + \frac{1}{q-1}(1 - \frac{m}{N})\} +$$

$$+ n \max_{i+1\leq j\leq N} q^{k_j+n+e}(\frac{2q}{q-1} + \lambda - 2k_j) < 0$$

Hence if we choose β large enough then from (1),...,(5) we get $-\infty < dgQ = dgN_{\beta+1} < 0$, which gives a contradiction to $I + Q = 0$ and $I \in \mathbb{F}_q[x]$. □

REMARK. Theorem 3.7 is our result A2 of §1. A special case of result A1 was proved in theorem 3.1 and A1 will be proved in the following theorem.

3.8. THEOREM. *Suppose* $\eta \neq 0$ *is a zero of the function*

$$f(t) = \sum_{k=0}^{\infty} c_k \frac{t^{q^k}}{F_k}$$

with $c_k \in \mathbb{F}_q$, $c_k \neq 0$ *for infinitely many* k. *Then* η *is transcendental over* $\mathbb{F}_q\{x\}$.

PROOF. The function $f(t)$ can be written also in the form

$$f(t) = \sum_{j=0}^{\infty} c_{v_j} \frac{t^{q^{v_j}}}{F_{v_j}}$$

with

$$c_{v_j} \in \mathbb{F}_q,\ c_{v_j} \neq 0 \qquad \text{for } j = 0,1,\ldots .$$

Let $\eta \neq 0$ be a zero of f, then from theorem 2.2 it follows that

$$dg\eta = v_\kappa + \frac{(v_\kappa - v_{\kappa-1})q^{v_{\kappa-1}}}{q^{v_\kappa} - q^{v_{\kappa-1}}},$$

where $v_\kappa > v_{\kappa-1}$ are non-negative integers determined by the coefficients of f: $c_{v_{\kappa-1}} \neq 0$, $c_{v_{\kappa-1}+1} = \ldots = c_{v_\kappa - 1} = 0$, $c_{v_\kappa} \neq 0$. Remark that $dg\eta > 0$.

Suppose η is algebraic, then $\alpha = \eta^{q^e}$ is separable with minimal-polynomial $P(t)$ of degree m. Let s be determined by $q^{s-1} < m \leq q^s$ and define $\lambda =: [dg\eta]$, where $[dg\eta]$ denotes the entier of the positive number $dg\eta$. Let

$$Q(t) = P(t)(t - x^{\lambda q^e})^{n-m},$$

where $n = q^s$. Then $Q(t)$ has the roots $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_m$; $\alpha_{m+1} = \ldots = \alpha_n = x^{\lambda q^e}$.

Define D and K_β as in the proof of theorem 3.1. Then $f^{q^e}(\eta) = 0$ and therefore

$$0 = K_\beta^{q^e} D^{q^{\beta+1}} \prod_{i=1}^{n} \left(\sum_{k=0}^{\infty} \frac{c_k}{F_k^{q^e}} \alpha_i^{q^k} \right) =$$

$$= K_\beta^{q^e} D^{q^{\beta+1}} \sum_{\nu=s}^{\infty} \sum_{\substack{k_1 \geq \dots \geq k_n \geq 0 \\ q^\nu \leq q^{k_1}+\dots+q^{k_n} < q^{\nu+1}}} \frac{c_{k_1} \cdots c_{k_n}}{F_{k_1}^{q^e} \cdots F_{k_n}^{q^e}} \sum_{(i_1,\dots,i_n)} \alpha_{i_1}^{q^{k_1}} \cdots \alpha_{i_n}^{q^{k_n}},$$

which sum is split into two parts $I + Q = 0$, where I is the sum over the terms with $\nu = s,s+1,\dots,\beta$. Then $I \in \mathbb{F}_q[x]$ which means either $\mathrm{dg}I \geq 0$ or $I = 0$. Every term of Q has the form

$$K_\beta^{q^e} D^{q^{\beta+1}} \frac{c_{k_1} \cdots c_{k_n}}{F_{k_1}^{q^e} \cdots F_{k_n}^{q^e}} \sum_{(i_1,\dots,i_n)} \alpha_{i_1}^{q^{k_1}} \cdots \alpha_{i_n}^{q^{k_n}}$$

with

$$k_1 \geq \dots \geq k_n \geq 0,\ q^\nu \leq q^{k_1}+\dots+q^{k_n} < q^{\nu+1},\ \nu \geq \beta + 1.$$

Let N_ν denote the term of Q where $q^{k_1}+\dots+q^{k_n} = q^\nu$ $(\nu\geq\beta+1)$. We shall prove:

(i) If β is chosen sufficiently large and such that $c_{\beta-s} \neq 0$ then $\mathrm{dg}N_{\beta+1} < 0$ and $N_{\beta+1} \neq 0$. Furthermore if $N_\nu \neq 0$ then for β sufficiently large

$$\mathrm{dg}N_{\nu+j} - \mathrm{dg}N_\nu < 0 \qquad \text{for } j = 1,2,\dots .$$

Hence for all $\nu > \beta + 1$ we have

$$\mathrm{dg}N_\nu < \mathrm{dg}N_{\beta+1}.$$

(ii) If N is a term of Q with $q^\nu < q^{k_1}+\dots+q^{k_n} < q^{\nu+1}$ $(\nu\geq\beta+1)$ and $N_\nu \neq 0$, then if β is chosen large enough we have

$$\mathrm{dg}N - \mathrm{dg}N_\nu < 0.$$

Hence for all terms N of Q, $N \neq N_{\beta+1}$ we have

$$\mathrm{dg}N < \mathrm{dg}N_{\beta+1},$$

which means

$$\mathrm{dg}Q = \mathrm{dg}N_{\beta+1} < 0.$$

Then it follows that

$$I = 0 \text{ and } Q = 0.$$

On the other hand since $N_{\beta+1} \neq 0$ and $dgQ = dgN_{\beta+1}$ we have $Q \neq 0$. Contradiction!

Proof of (i): The only possible solution for $k_1,\ldots,k_n$ in N_ν is $k_1 = \ldots = k_n = \nu - s$ and

$$N_\nu = K_\beta^{q^e} D^{q^{\beta+1}} \left(\frac{c_{\nu-s}}{F_{\nu-s}^{q^e}}\right)^n (\alpha_1\ldots\alpha_n)^{q^{\nu-s}}.$$

Hence if $c_{\nu-s} \neq 0$ we have $N_\nu \neq 0$ and

$$dgN_\nu = q^e dgK_\beta + q^{\beta+1} dgD - nq^e(\nu-s)q^{\nu-s} +$$
$$+ q^e(mq^e dg\eta + (n-m)q^e[dg\eta]) \qquad (*)$$

For $\nu = \beta + 1$ and $c_{\beta+1-s} \neq 0$ we get:

$$dgN_{\beta+1} \leq q^{\beta+e-n+1}\left[\beta(q^{n-1}+\ldots+q+1) + q^n(-\beta - 1 + s + \frac{dgD}{q^e} + \frac{m}{n}\, dg\eta + \frac{n-m}{n}[dg\eta])\right]$$

which is < 0 for all $\beta > \beta_1$ for which $c_{\beta+1-s} \neq 0$. Analogously for $j \geq 1$ and $c_{\nu-s} \neq 0$, $c_{\nu+j-s} \neq 0$ we find

$$dgN_{\nu+j} - dgN_\nu < 0,$$

which proves (i).

Proof of (ii): Now $q^\nu < q^{k_1}+\ldots+q^{k_n} < q^{\nu+1}$ and therefore $q^\nu = q^{k_1}+\ldots+q^{k_i}$ for a certain i, $1 \leq i < n$. We distinguish the two cases

(a) $i = q^\mu$ and $k_1 = \ldots = k_i = \nu - \mu$

(b) $i = q^{\mu_0} + q^{\mu_1}+\ldots+q^{\mu_r} - r$ with $\mu_0 \geq \ldots \geq \mu_r \geq 1$ and $\mu_0+\ldots+\mu_r \leq \nu$.

(iia) Now

$$N = K_\beta^{q^e} D^{q^{\beta+1}} \left(\frac{c_{\nu-\mu}}{F_{\nu-\mu}^{q^e}}\right)^i \frac{c_{k_{i+1}}\ldots c_{k_n}}{F_{k_{i+1}}^{q^e}\ldots F_{k_n}^{q^e}} \sum_{(j_1,\ldots,j_n)} (\alpha_{j_1}\ldots\alpha_{j_i})^{q^{\nu-\mu}} \alpha_{j_{i+1}}^{q^{k_{i+1}}}\ldots\alpha_{j_n}^{q^{k_n}},$$

which gives

$$dgN \le q^e dgK_\beta + q^{\beta+1}dgD - q^{\nu+e}(\nu-\mu) - k_{i+1}q^{k_{i+1}+e} - \ldots - k_n q^{k_n+e} +$$

$$+ q^e dg\eta(q^\nu + q^{k_{i+1}} + \ldots + q^{k_n}).$$

Suppose $N_\nu \neq 0$, then from (*) it follows that

$$dgN - dgN_\nu \le (\mu-s)q^{\nu+e} + q^{\nu+e}(dg\eta - [dg\eta])(1 - \frac{m}{n}) +$$

$$+ (n-i) \max_{i+1\le j\le n} q^{k_j+e}(dg\eta - k_j).$$

Since $\max_{x\ge 0} q^{x+e}(-x + dg\eta)$ does not depend on the choice of $k_{i+1},\ldots,k_n$ we conclude $dgN - dgN_\nu < 0$ for β large enough and $N_\nu \neq 0$.

(iib) Now

$$N = K_\beta^{q^e} D^{q^{\beta+1}} \sum_{k_1\ge\ldots\ge k_n\ge 0} \frac{c_{k_1}\ldots c_{k_n}}{F_{k_1}^{q^e}\ldots F_{k_n}^{q^e}} \sum_{(j_1,\ldots,j_n)} \alpha_{j_1}^{q^{k_1}} \ldots \alpha_{j_n}^{q^{k_n}},$$

where

$$\begin{cases} k_1 = \ldots = k_{q^{\mu_0}-1} & = \nu - \mu_0 \\ k_{q^{\mu_0}} = \ldots = k_{q^{\mu_0}+q^{\mu_1}-2} & = \nu - \mu_0 - \mu_1 \\ \vdots & \\ k_{q^{\mu_0}+\ldots+q^{\mu_{r-1}}-r+1} = \ldots = k_{q^{\mu_0}+\ldots+q^{\mu_r}-(r+1)} & = \nu - \mu_0 - \mu_1 - \ldots - \mu_r \\ k_{q^{\mu_0}+\ldots+q^{\mu_r}-r} & = \nu - \mu_0 - \ldots - \mu_r, \end{cases}$$

and therefore

$$dgN \le q^e dgK_\beta + q^{\beta+1}dgD - (\nu-\mu_0)q^{\nu+e} + q^{\nu+e}dg\eta +$$

$$+ (n-i) \max_{i+1\le j\le n} q^{k_j+e}(dg\eta - k_j).$$

Suppose $N_\nu \neq 0$ then

$$dgN - dgN_\nu \le (\mu_0 - s)q^{\nu+e} + q^{\nu+e}(dg\eta - [dg\eta])(1 - \frac{m}{n}) +$$
$$+ (n-i) \max_{i+1\le j\le n} q^{k_j+e}(dg\eta - k_j) < 0$$

if β is large enough.

From (i) and (ii) it follows that if $N \neq N_{\beta+1}$ then

$$dgN - dgN_{\beta+1} < 0.$$

We have proved now the contradiction $I = Q = 0$ and $dgQ = dgN_{\beta+1}$ $(N_{\beta+1} \neq 0)$, and therefore our assumption "η is algebraic" is false. $\square$

4. THE TRANSCENDENCY OF $\psi(\alpha)$ FOR ALGEBRAIC $\alpha \neq 0$

The function $f\colon \Phi \to \Phi$ given by the power series

$$f(t) = \sum_{i=0}^{\infty} a_i t^i$$

with $a_i \in \Phi$, which converges for all t with $dgt < R$ is called *linear* if

$$\begin{cases} f(t+u) = f(t) + f(u) \\ f(ct) = cf(t), \end{cases}$$

for all $t,u \in \Phi$ with $dgt < R$, $dgu < R$ and all $c \in \mathbb{F}_q$.

For linear functions we define for all t for which the involving series converge the *operators* Δ^r $(r=1,2,\ldots)$ by

$$\Delta f(t) = f(xt) - xf(t)$$
$$\Delta^r f(t) = \Delta^{r-1} f(xt) - x^{q^{r-1}} \Delta^{r-1} f(t), \quad r \ge 2.$$

For purpose of notation: $\Delta^\circ f(t) = f(t)$.

A power series $f\colon \Phi \to \Phi$ is called *entire* if f converges for all $t \in \Phi$.

For entire linear functions f we have an *"expansion formula"* (see L. CARLITZ [1]), namely:
for every $M \in \mathbb{F}_q[x]$ we have

$$f(Mt) = \sum_{\nu=0}^{dgM} \frac{\psi_\nu(M)}{F_\nu} \Delta^\nu f(t),$$

where F_ν is defined in def. 1.1 and

$$\psi_\nu(t) = \prod_{\substack{dgE<\nu \\ E\in \mathbb{F}_q[x]}} (t - E).$$

4.1. LEMMA. *Let* K *be a separable finite algebraic extension of* $\mathbb{F}_q\{x\}$ *of degree* h. *Let* $r,s \in \mathbb{N}$ *with* $0 < r < s$. *Then the system of linear equations*

$$\sum_{i=1}^{s} \alpha_{ki} X_i = 0 \qquad (k=1,\ldots,r),$$

where α_{ki} *are algebraic integers in* K *and*

$$a = \max_{k,i}(dg\alpha_{ki},0)$$

has a non-trivial solution $\{X_i\}_{i=1}^{s}$ *with* $X_i \in \mathbb{F}_q[x]$, *such that*

$$dgX_i < \frac{cs+ar}{s-r} \qquad (i=1,\ldots,s),$$

where c *is a positive constant only depending on the field* K.

PROOF. See [5], lemma 1. □

4.2. LEMMA. *Let*

$$\psi_\mu(t) := \prod_{\substack{E \\ dgE<\mu}} (t - E)$$

where $E \in \mathbb{F}_q[x]$. *Then*

$$dg \frac{\psi_\mu(A)}{F_\mu} < q^{m-1}$$

for all $A \in \mathbb{F}_q[x]$ *with* $dgA < m$ *and all* $\mu \in \{0,1,\ldots,m-1\}$.

PROOF. We may suppose that $dgA \geq \mu$, then

$$dg \frac{\psi_\mu(A)}{F_\mu} = \sum_{dgE<\mu} dg(A-E) - \mu q^\mu = (dgA-\mu)q^\mu.$$

Consider the function $f\colon \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f(y) = (\mathrm{dg}A-y)q^y.$$

Then

$$f'(y) = 0 \quad \text{for} \quad y = \mathrm{dg}A - \frac{1}{\ln q}.$$

Hence

$$f(y) \le \frac{1}{\ln q}\, q^{\mathrm{dg}A - \frac{1}{\ln q}} \quad \text{for all } y \ge 0.$$

For $q \ge 3$ we have

$$f(y) \le q^{\mathrm{dg}A-1} \qquad \text{for all } y \ge 0.$$

Hence

$$(\mathrm{dg}A-\mu)q^\mu \le q^{\mathrm{dg}A-1} < q^{m-1}, \quad \mu \in \{0,1,\dots,m-1\}.$$

For $q = 2$ we have $\ln 2 > \frac{1}{2}$ and therefore

$$f(y) < 2.q^{\mathrm{dg}A-2} = 2^{\mathrm{dg}A-1} < 2^{m-1} \quad \text{for all } \mu.$$

Hence for all $\mu \le \mathrm{dg}A$ we have

$$\mathrm{dg}\,\frac{\psi_\mu(A)}{F_\mu} < q^{m-1}.$$

For $\mu > \mathrm{dg}A$ we have $\psi_\mu(A) = 0$. □

4.3. THEOREM. *Let* $\alpha \in \Phi$, $\alpha \neq 0$ *and* $\alpha^{-1}\xi \notin \mathbb{F}_q\{x\}$. *Then* $\psi(\alpha)$ *is transcendental over* $\mathbb{F}_q\{x\}$.

REMARK. From this theorem it follows that $\psi(\alpha)$ is transcendental for algebraic $\alpha \neq 0$, which was already proved by L.I. WADE [8]. Now we give a different proof, which is a refinement of the methods used in [9], [4] and [5].

PROOF. Let $\alpha \in \Phi$, $\alpha \neq 0$ and $\alpha^{-1}\xi \notin \mathbb{F}_q\{x\}$. We may suppose that $\mathrm{dg}\alpha > \frac{q}{q-1}$; for if $\mathrm{dg}\alpha \le \frac{q}{q-1}$ then there exists $E \in \mathbb{F}_q[x]$ such that $\mathrm{dg}(E\alpha) > \frac{q}{q-1}$ and $(E\alpha)^{-1}\xi \notin \mathbb{F}_q\{x\}$, hence if we have proved the theorem for α with $\mathrm{dg}\alpha > \frac{q}{q-1}$ then $\psi(E\alpha)$ is transcendental. Using the expansionformula we get

$$\psi(E\alpha) = \sum_{\mu=0}^{dgE} (-1)^{\mu} \frac{\psi_{\mu}(E)}{F_{\mu}} \psi^{q^{\mu}}(\alpha)$$

and from the assumption $\psi(\alpha)$ algebraic we conclude $\psi(E\alpha)$ algebraic. Hence from now on $dg\alpha > \frac{q}{q-1}$.

Suppose $\psi(\alpha)$ is algebraic. Then there exists an $e \in \mathbb{N}$ such that $\psi^{q^e}(\alpha)$ is separable. Let $K = \mathbb{F}_q\{x\}(\psi^{q^e}(\alpha))$ and $h = [K: \mathbb{F}_q\{x\}]$. Define $d^*t = \max(dgt, 0)$. Define the function

$$L(t) := \sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} X_{ij} (d^*t)^j \psi^{iq^e}(\alpha t),$$

where k and l are positive integers which will be chosen later and the polynomials $X_{ij} \in \mathbb{F}_q[x]$ are to be defined by the following system of at most q^m equations in the q^{k+l} variables X_{ij}:

$$L(A) = 0 \quad \text{for } A \in \mathbb{F}_q[x],\ dgA < m.$$

The coefficients of X_{ij} in these equations are algebraic in K, since

$$0 = L(A) = \sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} X_{ij} (d^*A)^j \left\{ \sum_{\mu=0}^{m} (-1)^{\mu} \frac{\psi_{\mu}(A)}{F_{\mu}} \psi^{q^{\mu}}(\alpha) \right\}^{iq^e}.$$

In fact the coefficients of X_{ij} are polynomials in $\psi^{q^e}(\alpha)$ of degree $\leq q^{k+m}$ with coefficients in $\mathbb{F}_q\{x\}$. Let $\Gamma \in \mathbb{F}_q\{x\}$ be such that $\Gamma\psi^{q^e}(\alpha)$ is an algebraic integer of K. Let $c_1 = dg\Gamma$. Then $F_m^{q^{k+e}} \Gamma^{q^{k+m}} L(A) = 0$, $A \in \mathbb{F}_q[x]$, $dgA < m$ is a system of at most q^m linear equations in the X_{ij} with algebraic integers as coefficients; we write this system in the form

$$\sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} D_{ij} X_{ij} = 0.$$

Hence

$$\begin{aligned} dgD_{ij} &\leq q^{k+e}.mq^m + c_1 q^{k+m} + q^l m + q^{k+e}.q^{m-1} + q^{k+m+e}\max(0, dg\psi_{\mu}(\alpha)) \\ &\leq (m+c_2)q^{k+m+e} + mq^l, \end{aligned}$$

where c_2 is a positive real constant. Now put $k < \frac{1}{2}l$ and $m = k+l-1$, then

$$dgD_{ij} \leq (2m+c_3)q^{2l+e}$$

with $c_3 > 0$. We use lemma 4.1 with $r = q^m$, $s = q^{k+l}$, $a = \max d^*D_{ij}$, then we conclude that there exist

$$X_{ij} \in \mathbb{F}_q[x],\ i = 0,\dots,q^k-1;\ j = 0,\dots,q^l-1,$$

not all zero, such that

(1) $$\mathrm{dg}X_{ij} \le \frac{cq^{k+1} + (2m+c_3)q^{2l+e}.q^m}{q^{k+1} - q^m} = (2m+c_4)q^{2l+e},$$

where $c_4 > 0$.

Define

$$\mathcal{B}(\mu) := \{A + \alpha^{-1}\xi B \mid \mathrm{dg}B < \mathrm{dg}A < \mu;\ A,B \in \mathbb{F}_q[x] \text{ not both zero}\},$$

where

$$\xi = \lim_{k\to\infty} \frac{(x^q-x)^{\frac{q^k}{q-1}}}{L_k}$$

and therefore $\psi(B\xi) = 0$ for all $B \in \mathbb{F}_q[x]$. Hence for all $t \in \mathcal{B}(\mu)$ we have:

$$L(t) = L(A + \alpha^{-1}\xi B) =$$

$$= \sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} X_{ij}(d^*(A+\alpha^{-1}\xi B))^j\ \psi^{iq^e}(\alpha A + B\xi)$$

$$= \sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} X_{ij}(d^*(A+\alpha^{-1}\xi B))^j\ \psi^{iq^e}(\alpha A).$$

Since $\mathrm{dg}\alpha > \frac{q}{q-1}$ and $\mathrm{dg}\xi = \frac{q}{q-1}$ we have

$$\mathrm{dg}(A+\alpha^{-1}\xi B) = \max(\mathrm{dg}A, \mathrm{dg}\alpha^{-1}\xi B) = \mathrm{dg}A,$$

hence

(2) $$L(A+\alpha^{-1}\xi B) = \sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} X_{ij}(d^*A)^j\ \psi^{iq^e}(\alpha A).$$

This means for all $t \in \mathcal{B}(m)$ we have (since $\mathrm{dg}A < m$):

(3) $$L(t) = L(A+\alpha^{-1}\xi B) = L(A) = 0.$$

The number of polynomials B with $\mathrm{dg}B < \mathrm{dg}A$ and $\mathrm{dg}A = \nu$ is q^ν. Since

$\alpha^{-1}\xi \notin \mathbb{F}_q\{x\}$ we have

$$A_1 + \alpha^{-1}\xi B_1 = A_2 + \alpha^{-1}\xi B_2 \Longleftrightarrow A_1 = A_2 \text{ and } B_1 = B_2.$$

Hence the number of different elements $A + \alpha^{-1}\xi B$ with $dgB < dgA$ and $dgA = \nu$ is $q^{\nu}(q^{\nu+1}-q^{\nu})$, and therefore the number of elements of $\mathcal{B}(\mu)$ is

$$\sum_{\nu=0}^{\mu-1} q^{\nu}(q^{\nu+1}-q^{\nu}) = \frac{q^{2\mu}-1}{q+1} .$$

If we denote the number of elements of $B(\mu)$ by $N\mathcal{B}(\mu)$, then for $\mu > 1$ we have

$$q^{2\mu-2} < N\mathcal{B}(\mu) < q^{2\mu-1}. \tag{4}$$

Now let $\mu \geq m$. Define

$$\eta := \mu - k + 1 \tag{5}$$

then $\eta \geq 1$ and $\eta \geq 2k$.

Suppose $L(t) = 0$ for all $t \in \mathcal{B}(\mu)$. Let $D \in \mathbb{F}_q[x]$, $dgD = \mu$, then $D \in \mathcal{B}(\mu+1)$. The function

$$L(t) \Big/ \prod_{\mathcal{B}(\mu)} (t-A-\alpha^{-1}\xi B)$$

is an entire function, hence according to the maximum-modulus-principle (see [3]) we have

$$dg \frac{L(D)}{\prod_{\mathcal{B}(\mu)} (D-A-\alpha^{-1}\xi B)} \leq \max_{dgt=2\mu} dg\left(\frac{L(t)}{\prod_{\mathcal{B}(\mu)} (t-A-\alpha^{-1}\xi B)}\right) .$$

Since $\mu = dgD > \max(dgA, dg\alpha^{-1}\xi B)$ we have

$$dg \prod_{\mathcal{B}(\mu)} (D-A-\alpha^{-1}\xi B) = \mu . N\mathcal{B}(\mu)$$

and

$$\underset{dgt=2\mu}{dg} \prod_{\mathcal{B}(\mu)} (t-A-\alpha^{-1}\xi B) = 2\mu\, N\mathcal{B}(\mu).$$

Hence using (4) we have

$$dgL(D) \leq \max_{dgt=2\mu} dgL(t) - \mu N\mathcal{B}(\mu) < \max_{dgt=2\mu} dgL(t) - \mu q^{2\mu-2}.$$

From the definition of $L(t)$ we get by using (1) and (5):

(6)
$$\begin{aligned}\max_{dgt=2\mu} dgL(t) &\leq \max_{i,j} dgX_{ij} + 2\mu q^{1} + q^{k+e} \max_{dgt=2\mu} (dg\psi(\alpha t),0)\\ &\leq (2m+c_4)q^{2l+e} + 2\mu q^{l} + c_5 q^{k+e+2\mu}\\ &\leq 4\mu q^{2\eta+e} + c_6 q^{2\eta+e+3k-2}.\end{aligned}$$

Hence

$$dgL(D) \leq q^{2\eta+e}(4\mu + c_6 q^{3k-2} - \mu q^{2k-3-e}).$$

$L(D)$ is algebraic and $F_\mu^{q^{k+e}} \Gamma^{q^{\mu+k+e}} L(D)$ is an algebraic integer, hence

$$(N(F_\mu^{q^{k+e}} \Gamma^{q^{\mu+k+e}} L(D)) \in \mathbb{F}_q[x],$$

where $N(\beta) = \beta_1,\ldots,\beta_h$ where $\beta_1,\ldots,\beta_h$ are the conjugated elements of β in K if β is algebraic in K. Therefore

$$dg(N(F_\mu^{q^{k+e}} \Gamma^{q^{\mu+k+e}} L(D)) = h[\mu q^{k+e+\mu} + c_1 q^{\mu+k+e} + dgL(D)] =$$

$$\leq h[\mu q^{2\eta+e} + c_1 q^{2\eta+e} + dgL(D)] \leq hq^{2\eta+e}[\mu(5-q^{2k-3-e}) + c_7 q^{3k-2}].$$

Now first choose $k > k_0$ such that

$$5 - q^{2k-3-e} < 0$$

and then $l > l_0$ such that

$$l(5-q^{2k-3-e}) + c_7 q^{3k-2} < 0$$

then we must conclude

$$L(D) = 0.$$

According to (2) we now have for all $t \in \mathcal{B}(\mu+1)$:

(7)
$$L(D+\alpha^{-1}\xi B) = L(D) = 0.$$

Combining (3) and (7) we have

$$L(t) = 0 \text{ for all } t \in B(\mu),\ \mu = 1,2,\ldots .$$

We already have remarked that all these zeros of L(t) are different.

Now k and l are fixed under the above conditions. Since L is entire and not a polynomial we have (see [5]):

$$L(t) = \gamma_0 t \prod_{\eta \in \mathcal{B}(\nu)} \left(1 - \frac{t}{\eta}\right)^{\mu(\eta)} \prod_{\substack{\eta \notin \mathcal{B}(\nu) \\ \eta \neq 0 \\ \eta \text{ zero of } L}} \left(1 - \frac{t}{\eta}\right)^{\mu(\eta)}$$

with $\gamma_0 \in \Phi$, where $\mu(\eta)$ = multiplicity of the zero η and $\mu(\eta) \geq 1$. Furthermore

$$\max_{dgt=2\nu} dg \prod_{\substack{\eta \notin \mathcal{B}(\nu) \\ \eta \neq 0 \\ \eta \text{ zero of } L(t)}} \left(1 - \frac{t}{\eta}\right) \geq 0$$

and

$$\prod_{\eta \in \mathcal{B}(\nu)} \left(1 - \frac{t}{\eta}\right) = \frac{\prod_{\mathcal{B}(\nu)} (A+\alpha^{-1}\xi B - t)}{\prod_{\mathcal{B}(\nu)} (A+\alpha^{-1}\xi B)} .$$

From these formulae we get:

$$\max_{dgt=2\nu} dgL(t) \geq dg\gamma_0 + 2\nu + 2\nu N\mathcal{B}(\nu) - \sum_{\mathcal{B}(\nu)} dg(A+\alpha^{-1}\xi B)$$

$$\geq dg\gamma_0 + 2\nu + 2\nu N\mathcal{B}(\nu) - \nu N\mathcal{B}(\nu) \geq c_8 + 2\nu + \nu q^{2\nu-2}$$

where c_8 is a constant only depending on L(t). On the other hand from (6) we have

$$\max_{dgt=2\nu} dgL(t) \leq (2m+c_4)q^{2l+e} + c_5 q^{k+e+2\nu} + 2\nu q^{l},$$

where m, k and l are now fixed constants. Hence both inequalities for $\max_{dgt=2\nu} dgL(t)$ are contradictory for ν large enough, which proves our theorem. □

4.4. <u>LEMMA</u>. *If* $\alpha^{-1}\xi \in \mathbb{F}_q\{x\}$ *then* $\psi(\alpha)$ *is algebraic over* $\mathbb{F}_q\{x\}$.

PROOF. If $\alpha^{-1}\xi \in \mathbb{F}_q\{x\}$ then $\alpha = \frac{E}{F}.\xi$ with $E,F \in \mathbb{F}_q[x]$ and $(E,F) = 1$. [$(E,F) = 1$ means: E and F have no common factor.] If $\frac{E}{F} \in \mathbb{F}_q[x]$ then $\psi(\alpha) = 0$ (Cor. 2.9). Let $\frac{E}{F} \notin \mathbb{F}_q[x]$. Consider $\alpha_1 = \xi F^{-1}$. Then

$$0 = \psi(F.\alpha_1) = \sum_{\mu=0}^{dgF} (-1)^\mu \frac{\psi_\mu(F)}{F_\mu} .\psi^{q^\mu}(\alpha_1);$$

hence $\psi(\alpha_1)$ is a zero of an algebraic equation with coefficients in $\mathbb{F}_q\{x\}$ and therefore $\psi(\alpha_1)$ is algebraic. Now

$$\psi(\alpha) = \psi(E\alpha_1) = \sum_{\mu=0}^{dgE} (-1)^\mu \frac{\psi_\mu(E)}{F_\mu} \psi^{q^\mu}(\alpha)$$

is a rational combination of algebraic elements; hence $\psi(\alpha)$ is algebraic over $\mathbb{F}_q\{x\}$. □

Now we have proved the following theorem.

4.5. THEOREM. *If $\alpha \in \Phi$, $\alpha \neq 0$ then $\psi(\alpha)$ is transcendental over $\mathbb{F}_q\{x\}$ if and only if $\alpha^{-1}\xi \in \mathbb{F}_q\{x\}$.*

4.6. COROLLARY. *If $\alpha \in \Phi$, $\alpha \neq 0$ and α is algebraic, then $\psi(\alpha)$ is transcendental over $\mathbb{F}_q\{x\}$.*

PROOF. Since ξ is transcendental over $\mathbb{F}_q\{x\}$ (theorem 3.1) $\alpha^{-1}\xi$ is transcendental over $\mathbb{F}_q\{x\}$ and therefore $\alpha^{-1}\xi \notin \mathbb{F}_q\{x\}$, hence $\psi(\alpha)$ is transcendental over $\mathbb{F}_q\{x\}$. □

4.7. COROLLARY. *Let $\lambda(t): \sum_{j=0}^{\infty} \frac{t^{q^j}}{L_j}$ be defined for $\{t \in \Phi \mid dgt < \frac{q}{q-1}\}$. If $\alpha \in \Phi$, $\alpha \neq 0$, $dg\alpha < \frac{q}{q-1}$ and α is algebraic, then $\lambda(\alpha)$ is transcendental over $\mathbb{F}_q\{x\}$.*

PROOF. Since λ is the inverse function of ψ (see CARLITZ [1]) we have

$$\psi(\lambda(\alpha)) = \alpha.$$

Since $\alpha \neq 0$ and since λ has no zeros for $dgt < \frac{q}{q-1}$ we have $\lambda(\alpha) \neq 0$. Suppose $\lambda(\alpha)$ is algebraic, then according to cor. 4.6 $\psi(\lambda(\alpha))$ is transcendental over $\mathbb{F}_q\{x\}$. This contradicts our assumption that α is algebraic. □

4.8. THEOREM. *Suppose $\alpha \in \Phi$, $\alpha \neq 0$ and α is algebraic over $\mathbb{F}_q\{x\}$. Define*

$$f(t) := \sum_{j=0}^{\infty} c_j \frac{t^{q^j}}{F_j},$$

where $c_j \in \mathbb{F}_q$ *and* $\exists c \in \mathbb{F}_q$, $c \neq 0$ *such that* $c_{j+1} = c.c_j$, $j = 0,1,2,\ldots$.
Then $f(\alpha)$ *is transcendental over* $\mathbb{F}_q\{x\}$.

<u>PROOF</u>. Suppose $f(\alpha)$ is algebraic and let $f^{q^e}(\alpha)$ be separable. Let $\Gamma \in \mathbb{F}_q[x]$ be such that $\Gamma f^{q^e}(\alpha)$ is an algebraic integer. $K = \mathbb{F}_q\{x\}(f^{q^e}(\alpha))$ and $[K: \mathbb{F}_q\{x\}] = h$. As in theorem 4.3 we define

$$L(t) := \sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} X_{ij}(d^*t)^j f^{iq^e}(\alpha t),$$

where k and $l \in \mathbb{N}$, $k < \frac{1}{2}l$ will be chosen later and $X_{ij} \in \mathbb{F}_q[x]$ are defined by the following system of equations

$$L(A) = 0 \quad \text{for } A \in \mathbb{F}_q[x],\ \mathrm{dg}A < m,$$

where $m = k + l - 1$. By the same arguments as used in theorem 4.3 we find: the X_{ij} are not all zero and

$$\mathrm{dg}X_{ij} \leq (2m + c_4)q^{2l+e}$$

where c_4 is a positive real constant which does not depend on m, l and k.

There exists a minimal $m_0 \in \mathbb{N} \cup \{0\}$ such that

$$\mathrm{dg}\alpha > \frac{1}{q-1} - m_0.$$

Now we want k and l to be such that $m > m_0$.

Define

$$\mathcal{B}(\mu) = \left\{ A + \alpha^{-1}\beta \;\middle|\; \begin{array}{l} A \in \mathbb{F}_q[x],\ \beta \text{ a zero of } f(t),\ A \text{ and } B \text{ not} \\ \text{both zero; } \mathrm{dg}A < \mu,\ \mathrm{dg}\beta < \mathrm{dg}A + \frac{1}{q-1} - m_0. \end{array} \right\}$$

For all $t \in \mathcal{B}(\mu)$ we have

$$L(t) = L(A+\alpha^{-1}\beta) = \sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} X_{ij}(d^*(A+\alpha^{-1}\beta))^j f^{iq^e}(\alpha A+\beta) =$$

$$= \sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} X_{ij}(d^*(A+\alpha^{-1}\beta))^j f^{iq^e}(\alpha A).$$

Since $\mathrm{dg}\beta < \mathrm{dg}A + \frac{1}{q-1} - m_0$ we have $d^*(A+\alpha^{-1}\beta) = \mathrm{dg}A$. Therefore for all

$t \in \mathcal{B}(\mu)$ we have:

$$L(t) = \sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} x_{ij} (d^*A)^j \, f^{iq^e}(\alpha A).$$

Hence

$$L(t) = 0, \ \forall t \in \mathcal{B}(m).$$

According to cor. 2.9 the number of zeros β of $f(t)$ with $dg\beta = \frac{1}{q-1} + k$ is $q^k - q^{k-1}$, $k = 1,2,\ldots$; hence the number of zeros β of $f(t)$ with $dg\beta < dgA + \frac{1}{q-1} - m_0$ with $dgA = \nu \geq m_0$ is $q^{\nu-m_0}$. The number of polynomials A with $dgA = \nu$ is $q^{\nu+1} - q^{\nu}$. Since $f(t)$ is linear all zeros of $f(t)$ are different, for if $f(\beta) = 0$ and $dg\beta = \frac{1}{q-1} + k$, then $c\beta + \beta_1$ where $f(\beta_1) = 0$ and $dg\beta_1 < \frac{1}{q-1} + k$ and $c \in \mathbb{F}_q$, $c \neq 0$ is also a zero of f and $dg(c\beta+\beta_1) = \frac{1}{q-1} + k$. Since $c \neq 0$ and $\beta_1 \neq 0$ we have $c\beta + \beta_1 \neq \beta$. The number of different zeros $c\beta + \beta_1$ with $dg\beta = \frac{1}{q-1} + k$, $c \in \mathbb{F}_q$, $c \neq 0$ and $dg\beta_1 < \frac{1}{q-1} + k$ is $q^k - q^{k-1}$. Hence all zeros of $f(t)$ are different.

Since α is algebraic and according to theorem 3.1 a zero $\beta \neq 0$ of $f(t)$ is transcendental, we have $\alpha^{-1}\beta$ is transcendental for every zero $\beta \neq 0$ of $f(t)$. Suppose

$$A_1 + \alpha^{-1}\beta_1 = A_2 + \alpha^{-1}\beta_2,$$

then

$$A_1 - A_2 = \alpha^{-1}(\beta_2-\beta_1).$$

Since $\beta_2 - \beta_1$ is a zero of $f(t)$ it follows that $\alpha^{-1}(\beta_2-\beta_1)$ is transcendental unless $\beta_1 = \beta_2$. Hence $A_1 = A_2$ and $\beta_1 = \beta_2$. This means that all elements $A + \alpha^{-1}\beta$ of $\mathcal{B}(\mu)$ are different. Therefore the number of elements of $\mathcal{B}(\mu)$ is

$$\sum_{\nu=m_0}^{\mu-1} q^{\nu-m_0}(q^{\nu+1}-q^{\nu}) + q^{m_0} - 1 = \frac{q^{2\mu-m_0}+q^{m_0+1}}{q+1} - 1.$$

If we denote the number of elements of $\mathcal{B}(\mu)$ by $N\mathcal{B}(\mu)$ then for $\mu > m_0$ we have

$$q^{2\mu-m_0-2} < N\mathcal{B}(\mu) < q^{2\mu-1}.$$

Now we proceed in the same way as in theorem 4.3, which leads to the choice of k such that $q^{2k-3-m_0-e} > 5$ and then l such that $l(5-q^{2k-3-m_0-3}) + c_7q^{3k-2} < 0$ where c_7 is the same constant as in theorem 4.3. With this choice of k and l we conclude that $L(t) = 0$ for all $t \in \mathcal{B}(\mu)$, $\mu = 1,2,\ldots$.

This gives us a productrepresentation for L(t) from which it follows that

$$\max_{dgt=2\nu} dgL(t) \geq c_8 + 2\nu + \nu q^{2\nu-m_0-2},$$

and on the other hand from the definition of L(t) we get

$$\max_{dgt=2\nu} dgL(t) \leq (2m+c_4)q^{2l+e} + c_5q^{k+e+2\nu} + 2\nu q^{l},$$

which gives for ν large enough the desired contradiction. Hence our assumption that $f(\alpha)$ is algebraic is false. □

REMARK. In the proof of theorem 4.8 we use the expansionformula for f(t) to determine the X_{ij}, namely since $c_{j+1} = c.c_j$ we get

$$f(\alpha A) = \sum_{\mu=0}^{dgA} \frac{\psi_\mu(A)}{F_\mu} \Delta^\mu f(\alpha) = \sum_{\mu=0}^{dgA} \frac{\psi_\mu(A)}{F_\mu} c^\mu f^{q^\mu}(\alpha)$$

from which it follows that the system of linear equations

$$L(A) = 0, \qquad A \in \mathbb{F}_q[x],\ dgA < m$$

in the X_{ij} has algebraic coefficients. Hence it does not seem possible to generalize theorem 4.8 to all functions of the form

$$g(t) = \sum_{j=0}^{\infty} c_j \frac{t^{q^j}}{F_j},\ c_j \in \mathbb{F}_q,\ c_j \neq 0 \text{ for infinitely many } j,$$

without a different method.

L.I. WADE proved in [8] that for $E \in \mathbb{F}_q[x]$, g(E) is transcendental over $\mathbb{F}_q\{x\}$ as a consequence of his theorem 3.2:

"If $B_0, B_1, \ldots$ satisfy the conditions

(i) $B_k \in \mathbb{F}_q[x]$

(ii) $B_k \neq 0$ for infinitely many k

(iii) $dgB_k \leq (q-1)(k-1)q^{k-1} - b_kq^k$ for all sufficiently large k, where $b_k \to \infty$ as $k \to \infty$

then $\sum_{k=0}^{\infty} \frac{B_k}{F_k}$ is transcendental over $\mathbb{F}_q\{x\}$".

In a forthcoming paper we shall discuss a more general result of this type.

5. TRANSCENDENCE PROPERTIES OF BESSELFUNCTIONS IN ALGEBRAIC NONZERO POINTS

In [2] L. CARLITZ introduced the following function:

$$J_n(t) = \sum_{r=0}^{\infty} (-1)^r \frac{t^{q^{n+r}}}{F_{n+r}F_r^{q^n}}, \quad n \in \mathbb{Z},$$

where $(F_{-n})^{-1} := 0$ for $n = 1,2,\ldots$. For these functions, which we shall call the Carlitz-Bessel-functions, he proved among other things the following relations for all $n \in \mathbb{Z}$:

$$J_{-n}(t) = (-1)^n \{J_n(t)\}^{q^{-n}}$$

$$\Delta^r J_n(t) = J_{n-r}^{q^r}(t), \qquad r = 0,1,2,\ldots$$

$$J_{n+1}(t) - (x^{q^n}-x)J_n(t) + J_{n-1}^q(t) = 0.$$

Using these relations we proved in [4], lemma 3.1 the following lemma:

5.1. <u>LEMMA</u>. *$J_{2n}(t)$ and $J_{2n+1}(t)$ are linear polynomials in $J_0(t)$ and $\Delta J_0(t)$ of degree q^n with coefficients in $\mathbb{F}_q[x]$ of degree $< q^{2n}$ respectively $< q^{2n+1}$.*

5.2. <u>LEMMA</u>. *Let $n \in \mathbb{N} \cup \{0\}$, then $J_n(t)$ has a zero of order q^n in $t = 0$ and has $q^{k+1} - q^k$ different zeros β with $dg\beta = n + 2k + \frac{2q}{q-1}$. Each of these zeros β has order q^n.*

<u>PROOF</u>. According to cor. 2.9 $J_n(t)$ has a zero of order q^n in $t = 0$ and has $q^{n+k+1} - q^{n+k}$ zeros β with

$$dg\beta = n + 2(k+1) + \frac{2}{q-1}, \qquad k = 0,1,\ldots$$

and $J_n(t)$ does not have any other zeros.

Let β be a zero of $J_n(t)$ with $dg\beta = n + 2(k+1) + \frac{2}{q-1}$ then the elements $c\beta + \beta^*$ with $c \in \mathbb{F}_q$, $c \neq 0$ and β^* a zero of $J_n(t)$ with $dg\beta^* < dg\beta$ are all different. If $k = 0$ the number of different elements $c\beta + \beta^*$ is $q-1$. From this it follows inductively for arbitrary $k > 0$ that the number of different elements $c\beta + \beta^*$ is $q^{k+1} - q^k$. Hence the number of different zeros β with $dg\beta = n + 2(k+1) + \frac{2}{q-1}$ is $q^{k+1} - q^k$.

Since $J_{-n}(t) = (-1)^n\{J_n(t)\}^{q^{-n}}$ it follows that if β is a zero of $J_n(t)$ then $J_{-n}(\beta) = 0$. Hence

$$J_n(t) = (-1)^n\{J_{-n}(t)\}^{q^n}$$

and therefore β is a zero of $J_n(t)$ with multiplicity $\geq q^n$. Since the total number of zeros β with $dg\beta = n + 2k + \frac{2q}{q-1}$ is $q^{n+k+1} - q^{n+k}$ the zero β has multiplicity exactly q^n. □

5.3. THEOREM. *If $\alpha \in \Phi$, $\alpha \neq 0$ and α algebraic over $\mathbb{F}_q\{x\}$ then $J_n(\alpha)$ and $\Delta J_n(\alpha)$ cannot be both algebraic over $\mathbb{F}_q\{x\}$.*

PROOF. It is sufficient to consider only the case $n \geq 0$; for if $n < 0$ then from the assumption $J_n(\alpha)$ and $\Delta J_n(\alpha)$ are both algebraic it follows that since

$$J_{-n}(\alpha) = (-1)^n\{J_n(\alpha)\}^{q^{-n}},$$

$J_{-n}(\alpha)$ is algebraic and since

$$\begin{aligned}\Delta J_{-n}(\alpha) &= J_{-n}(x\alpha) - xJ_{-n}(\alpha) = \\ &= (-1)^n\{\Delta J_n(\alpha)\}^{q^{-n}} + (-1)^n(x^{q^{-n}}-x)\{J_n(\alpha)\}^{q^{-n}},\end{aligned}$$

also $\Delta J_{-n}(\alpha)$ is algebraic. Hence we suppose from now on $n \geq 0$.

Suppose $J_n(\alpha)$ and $\Delta J_n(\alpha)$ are both algebraic. Then $\exists e \in \mathbb{N}$ such that $J_n^{q^e}(\alpha)$ and $(\Delta J_n(\alpha))^{q^e}$ are separable and they generate a normal algebraic extension K of $\mathbb{F}_q\{x\}$ of degree h.

Let $\Gamma \in \mathbb{F}_q\{x\}$ be such that $\Gamma J_n^{q^e}(\alpha)$ and $\Gamma(\Delta J_n(\alpha))^{q^e}$ are algebraic integers in K. Define

$$L(t) := \sum_{i=0}^{q^k-1} \sum_{j=o}^{q^l-1} x_{ij}(d^*t)^j\, J_n^{iq^e}(\alpha t),$$

where k and $l \in \mathbb{N}$ are to be chosen later such that $k < \frac{1}{2}$ and the x_{ij} are to be determined by the following system of equations

$$L(A) = 0 \qquad \text{for } A \in \mathbb{F}_q[x],\ dgA < m,$$

where $m := k + l - 1$. This means

$$\sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} X_{ij} (d^*A)^j \left(\sum_{\mu=0}^{dgA} \frac{\psi_\mu(A)}{F_\mu} \Delta^\mu J_n(\alpha) \right)^{iq^e} = 0, \quad dgA < m.$$

Using the relations for $J_n(t)$ this system can be written as

$$\sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} X_{ij} (d^*A)^j \left(\sum_{\mu=0}^{dgA} \frac{\psi_\mu(A)}{F_\mu} J_{n-\mu}^{q^\mu}(\alpha) \right)^{iq^e} = 0, \quad dgA < m,$$

or

$$\sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} X_{ij} (d^*A)^j \left(\sum_{\mu=0}^{n} \frac{\psi_\mu(A)}{F_\mu} J_{n-\mu}^{q^\mu}(\alpha) + \sum_{\mu=n+1}^{dgA} (-1)^\mu \frac{\psi_\mu(A)}{F_\mu} J_{\mu-n}(\alpha) \right)^{iq^e} = 0, \ dgA < m.$$

It now follows from this and lemma 5.1 that the coefficients of X_{ij} in this system of equations are polynomials in $J_0^{q^e}(\alpha)$ and $(\Delta J_0(\alpha))^{q^e}$ of degree $\le q^{[\frac{m+n}{2}]+1}(q^k-1)$ with coefficients in $\mathbb{F}_q\{x\}$. Hence if $m > 2n$ and since $m = k + l - 1$ using lemma 5.1 we have:

$$F_m^{q^{k+e}} \Gamma^{q^{2l}} L(A) = \sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} D_{ij} X_{ij} = 0, \quad dgA < m,$$

is a system of at most q^m linear equations in the X_{ij} with algebraic integers as coefficients, where

$$dgD_{ij} \leqq mq^{k+m+e} + c_0 q^{2l} + q^{k+e}(q^{m-1} + q^{m-n} + c_1 q^{1+[\frac{m+n}{2}]})$$

with $c_0 = dg\Gamma$ and $c_1 = \max(dgJ_0(\alpha), dg\Delta J_0(\alpha), 0)$, hence

$$dgD_{ij} \le q^{2l+e}(2m + c_2)$$

where $c_2 > 0$. Using lemma 4.1 we conclude that there exist

$$X_{ij} \in \mathbb{F}_q[x],\ i = 0,1,\ldots,q^k-1;\ j = 0,1,\ldots,q^l-1,$$

not all zero such that

$$dgX_{ij} \le \frac{cq^{k+1}+(2m+c_2)q^{2l+e}\cdot q^m}{q^{k+1}-q^m} = (2m+c_3)q^{2l+e},$$

with $c_3 > 0$.

Let $m_0 \in \mathbb{N} \cup \{0\}$ be minimal such that

$$dg\alpha > n + \frac{2}{q-1} - m_0.$$

We later choose k such that $k > m_0$. Define

$$\mathcal{B}(\mu) := \left\{ A + \alpha^{-1}\beta \;\middle|\; \begin{array}{l} A \in \mathbb{F}_q[x],\ \beta \text{ a zero of } J_n(t),\ A \text{ and } \beta \text{ not} \\ \text{both } 0;\ dgA < \mu,\ dg\beta < dgA + n + \frac{2}{q-1} - m_0 \end{array} \right\}.$$

For all $t \in \mathcal{B}(\mu)$ we have

$$L(t) = L(A+\alpha^{-1}\beta) = \sum_{i=0}^{q^k-1} \sum_{j=0}^{q^l-1} X_{ij}(d^*(A+\alpha^{-1}\beta))^j J_0^{iq^e}(\alpha A+\beta).$$

Now we have

$$J_0(\alpha A+\beta) = J_0(\alpha A)$$

and since

$$dg\alpha^{-1}\beta < -dg\alpha + dgA + n + \frac{2}{q-1} - m_0 < dgA,$$

$$d^*(A+\alpha^{-1}\beta) = d^*(A),$$

hence

$$L(A+\alpha^{-1}\beta) = L(A)$$

which gives

$$L(t) = 0 \quad \text{for } t \in \mathcal{B}(m).$$

According to lemma 5.2 $J_n(t)$ has a zero of order q^n in $t = 0$ and $J_n(t)$ has $q^{k+1} - q^k$ different zeros β with

$$dg\beta = n + 2(k+1) + \frac{2}{q-1}$$

each of order q^n, $k = 1,2,\ldots$. Since $\alpha \neq 0$ is algebraic and the zeros $\beta \neq 0$ of $J_n(t)$ are transcendental (th. 3.7), the number of different elements $A + \alpha^{-1}\beta$ of $\mathcal{B}(\mu)$ with $dgA = \nu \ge m_0$ is

$$q^{\nu-m_0}(q^{\nu+1}-q^{\nu}).$$

Therefore the total number of elements of $\mathcal{B}(\mu)$ is

$$N\mathcal{B}(\mu) = \sum_{\nu=m_0}^{\mu-1} q^{\nu-m_0}(q^{\nu+1}-q^{\nu}) + q^{m_0} - 1$$

which gives for $\mu \geq m$:

$$q^{2\mu-2} < N\mathcal{B}(\mu) < q^{2\mu-1}.$$

Define $\eta := \mu - k + 1$ and suppose $L(t) = 0$ for $t \in \mathcal{B}(\mu)$. The function

$$L(t) \Big/ \prod_{\mathcal{B}(\mu)} (t-A-\alpha^{-1}\beta)$$

is an entire function and according to the maximum modulus principle we have

$$\operatorname{dg}L(D) \leq \sum_{\mathcal{B}(\mu)} \operatorname{dg}(D-A-\alpha^{-1}\beta) + \max_{\operatorname{dg}t=2\mu} \operatorname{dg}\left(\frac{L(t)}{\prod_{\mathcal{B}(\mu)} (t-A-\alpha^{-1}\beta)}\right)$$

$$\leq \max_{\operatorname{dg}t=2\mu} \operatorname{dg}L(t) - \mu N\mathcal{B}(\mu).$$

Furthermore

$$\max_{\operatorname{dg}t=2\mu} \operatorname{dg}L(t) \leq \max_{i,j} \operatorname{dg}X_{ij} + 2\mu q^{l} + q^{k+e} \max_{\operatorname{dg}t=2\mu} (\operatorname{dg}J_n(\alpha t), 0).$$

Since

$$J_n(\alpha t) = \sum_{r=0}^{\infty} (-1)^r \frac{(\alpha t)^{q^{n+r}}}{F_{n+r}F_r^{q^n}},$$

we have

$$\max_{\operatorname{dg}t=2\mu} \operatorname{dg}J_n(\alpha t) \leq \max_{r\geq 0} q^{n+r}(\operatorname{dg}\alpha+2\mu-n-2r) \leq$$

$$\leq q^{n+\mu-[\frac{m_0}{2}]}(1 + \frac{2}{q-1}) = c_4 q^{\mu+n}$$

where $c_4 > 0$ only depends on α. Hence

$$dgL(D) \leqq (2m+c_3)q^{2l+e} + 2\mu q^{l} + c_4 q^{k+e+\mu+n} - q^{2\mu-2}$$

$$\leqq q^{2\eta+e}(4\mu+c_5 q^{2k-1+n}-\mu q^{2k-3-e}).$$

Since $L(D)$ is algebraic it follows from the definition of L and lemma 5.1 that

$$F_{\mu}^{q^{k+e}} \Gamma^{q^{2\eta}} L(D)$$

is an algebraic integer G, and

$$N(G) \leqq h[q^{k+e+\mu}\mu+c_0 q^{2\eta}+q^{2\eta+e}(4\mu+c_5 q^{2k-1+n}-\mu q^{2k-3-e})]$$

$$\leqq hq^{2\eta+e}[\mu(5-q^{2k-3-e})+c_6 q^{2k-1+n}]$$

with $c_6 > 0$. Now choose k such that $5 - q^{2k-3-e} < 0$ and $k > m_0$ and then l such that $l(5-q^{2k-3-e}) + c_6 q^{2k-1+n} < 0$ then $L(D) = 0$.

Now the integers k, l and m are fixed. We have $L(t) = 0$ for all $t \in B(\mu)$, $\mu \geq m$.

Since $L(t)$ is an entire function we have

$$L(t) = \gamma_0 t^{q^n} \prod_{\eta\in\mathcal{B}(\nu)} (1 - \frac{t}{\eta})^{\mu(\eta)} \prod_{\substack{\eta\notin\mathcal{B}(\nu)\\ \eta\neq 0\\ \eta \text{ zero of } L(t)}} (1 - \frac{t}{\eta})^{\mu(\eta)},$$

with $\gamma_0 \in \Phi$, where $\mu(\eta) \geq 1$ is the multiplicity of the zero η of $L(t)$.

Let ν_0 be the minimum of the degrees of the zeros $\neq 0$ of $L(t)$, then

$$\max_{dgt=2\nu} \prod_{\substack{\eta\notin\mathcal{B}(\nu)\\ \eta\neq 0}} (1 - \frac{t}{\eta})^{\mu(\eta)} \geq \max_{dgt=\frac{\nu_0}{2}} \prod_{\substack{\eta\notin\mathcal{B}(\nu)\\ \eta\neq 0}} (1 - \frac{t}{\eta})^{\mu(\eta)} = 0.$$

Therefore

$$\max_{dgt=2\nu} dgL(t) \geq dg\gamma_0 + 2\nu q^n + 2\nu N\mathcal{B}(\nu) - \sum_{\mathcal{B}(\nu)} dg(A+\alpha^{-1}\beta)$$

$$\geq dg\gamma_0 + 2\nu q^n + \nu N\mathcal{B}(\nu) \geq c_7 + 2\nu q^n + \nu q^{2\nu-2}.$$

On the other hand

$$\max_{dgt=2\nu} dgL(t) \leq (2m+c_3)q^{2l+e} + 2\nu q^{l} + c_4 q^{k+e+n+\nu}$$

and both inequalities are contradictory for ν sufficiently large. Hence at least one of the elements $J_n(\alpha)$, $\Delta J_n(\alpha)$ is transcendental over $\mathbb{F}_q\{x\}$. □

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